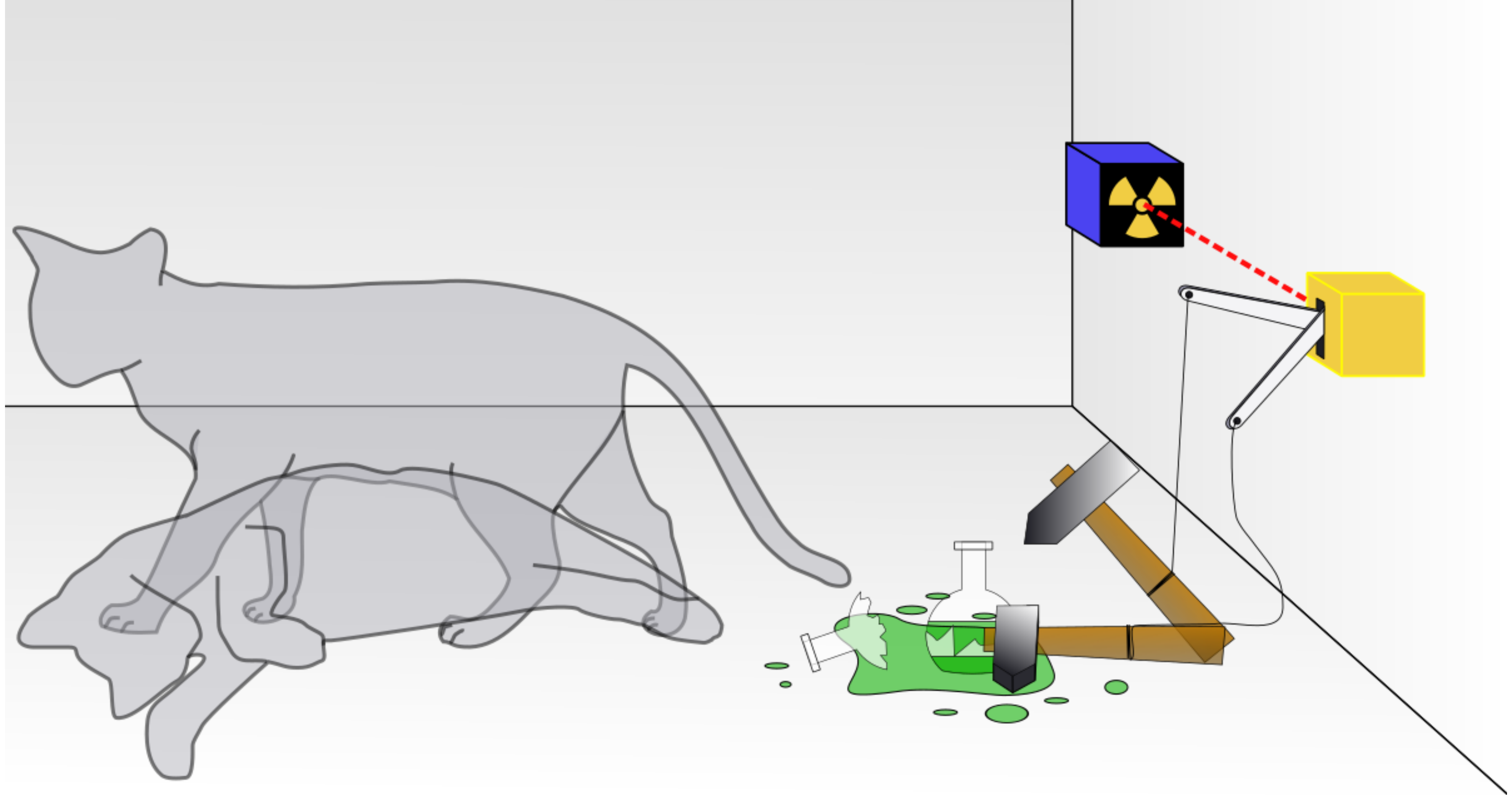


Lecture 07

Postulates of Quantum Mechanics

Expectation values of p and p^2 for ground state of Infinite well

$$\langle p \rangle = \int_0^L \left(\sqrt{\frac{2}{L}} \sin\frac{nx}{L}\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(\sqrt{\frac{2}{L}} \sin\frac{nx}{L}\right) dx$$

$$= \frac{\hbar}{i}\frac{2}{L}\frac{\pi}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{\pi x}{L}\, dx = 0$$

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi = -\hbar^2\frac{\partial^2\psi}{\partial x^2} = -\hbar^2\left(-\frac{\pi^2}{L^2}\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\right)$$

$$= +\frac{\hbar^2\pi^2}{L^2}\psi$$

$$\langle p^2 \rangle = \frac{\hbar^2\pi^2}{L^2}\int_0^L \psi^*\psi\, dx = \frac{\hbar^2\pi^2}{L^2}$$

Postulate 1: State of a system

The state of any physical system is specified, at each time t, by a state vector $|\psi(t)\rangle$ in a Hilbert space $\mathcal{H}$; $|\psi(t)\rangle$ contains (and serves as the basis to extract) all the needed information about the system. Any superposition of state vectors is also a state vector.

Schrodinger Formulation and Heisenberg Formulation

Postulate 2: Observables and operators
To every physically measurable quantity A, called an observable or dynamical variable, there corresponds a linear Hermitian operator $\hat{A}$ whose eigenvectors form a complete basis.

Observable	Corresponding operator
$\vec{r}$	$\hat{\vec{R}}$
$\vec{p}$	$\hat{\vec{P}} = -i\hbar\vec{\nabla}$
$T = \frac{p^2}{2m}$	$\hat{T} = -\frac{\hbar^2}{2m}\nabla^2$
$E = \frac{p^2}{2m} + V(\vec{r}, t)$	$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \hat{V}(\hat{\vec{R}}, t)$
$\vec{L} = \vec{r} \times \vec{p}$	$\hat{\vec{L}} = -i\hbar\hat{\vec{R}} \times \vec{\nabla}$

Postulate 3: Measurements and eigenvalues of operators

The measurement of an observable A may be represented formally by the action of $\hat{A}$ on a state vector $|\psi(t)\rangle$. The only possible result of such a measurement is one of the eigenvalues a_n (which are real) of the operator $\hat{A}$. If the result of a measurement of A on a state $|\psi(t)\rangle$ is a_n, the state of the system *immediately after* the measurement changes to $|\psi_n\rangle$:

$$\hat{A}|\psi(t)\rangle = a_n|\psi_n\rangle,$$

where $a_n = \langle\psi_n|\psi(t)\rangle$. **Note:** a_n is the component of $|\psi(t)\rangle$ when projected[1] onto the eigenvector $|\psi_n\rangle$.

Postulate 4: Probabilistic outcome of measurements

- **Discrete spectra**: When measuring an observable A of a system in a state $|\psi\rangle$, the probability of obtaining one of the nondegenerate eigenvalues a_n of the corresponding operator $\hat{A}$ is given by

$$P_n(a_n) = \frac{|\langle\psi_n|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{|a_n|^2}{\langle\psi|\psi\rangle},$$

  where $|\psi_n\rangle$ is the eigenstate of $\hat{A}$ with eigenvalue a_n. If the eigenvalue a_n is m-degenerate, P_n becomes

$$P_n(a_n) = \frac{\sum_{j=1}^{m} |\langle\psi_n^j|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \frac{\sum_{j=1}^{m} |a_n^{(j)}|^2}{\langle\psi|\psi\rangle}.$$

  The act of measurement changes the state of the system from $|\psi\rangle$ to $|\psi_n\rangle$. If the system is already in an eigenstate $|\psi_n\rangle$ of $\hat{A}$, a measurement of A yields with certainty the corresponding eigenvalue a_n: $\hat{A}|\psi_n\rangle = a_n|\psi_n\rangle$.

- **Continuous spectra:** The relation (3.2), which is valid for discrete spectra, can be extended to determine the probability density that a measurement of $\hat{A}$ yields a value between a and $a + da$ on a system which is initially in a state $|\psi\rangle$:

$$\frac{dP(a)}{da} = \frac{|\psi(a)|^2}{\langle\psi|\psi\rangle} = \frac{|\psi(a)|^2}{\int_{-\infty}^{+\infty} |\psi(a')|^2 da'};$$

  for instance, the probability density for finding a particle between x and $x + dx$ is given by $dP(x)/dx = |\psi(x)|^2/\langle\psi|\psi\rangle$.

Postulate 5: Time evolution of a system

The time evolution of the state vector $|\psi(t)\rangle$ of a system is governed by the time-dependent *Schrödinger equation*

$$i\hbar\frac{\partial|\psi(t)\rangle}{\partial t} = \hat{H}|\psi(t)\rangle,$$

where $\hat{H}$ is the Hamiltonian operator corresponding to the total energy of the system.

Connecting Quantum to Classical Mechanics

Poisson Brackets and Commutators

$$\{A, B\} = \sum_j \left(\frac{\partial A}{\partial q_j} \frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial B}{\partial q_j} \right).$$

$$\partial q_j / \partial p_k = 0, \partial p_j / \partial q_k = 0;$$

$$\{q_j, q_k\} = \{p_j, p_k\} = 0, \qquad \{q_j, p_k\} = \delta_{jk}.$$

$$\{A,\ B\} = -\{B,\ A\}$$

$$\{A,\ \alpha B + \beta C + \gamma D + \cdots\} = \alpha\{A,\ B\} + \beta\{A,\ C\}$$

$$\{A,\ B\}^* = \{A^*,\ B^*\}$$

$$\{A,\ BC\} = \{A,\ B\}C + B\{A,\ C\},$$

$$\{A, \{B,\ C\}\} + \{B,\ \{C,\ A\}\} + \{C,\ \{A,\ B\}\} = 0$$

$$\{A,\ B^n\} = nB^{n-1}\{A,\ B\}, \qquad \{A^n,\ B\} = nA^{n-1}\{A,\ B\}$$

The total time derivative of a dynamical variable A is given by

$$\frac{dA}{dt} = \sum_j \left(\frac{\partial A}{\partial q_j} \frac{\partial q_j}{\partial t} + \frac{\partial A}{\partial p_j} \frac{\partial p_j}{\partial t} \right) + \frac{\partial A}{\partial t} = \sum_j \left(\frac{\partial A}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial A}{\partial p_j} \frac{\partial H}{\partial p_j} \right) + \frac{\partial A}{\partial t};$$

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j},$$

$$\frac{dA}{dt} = \{A, H\} + \frac{\partial A}{\partial t}.$$

Time Evolution of Expectation Values

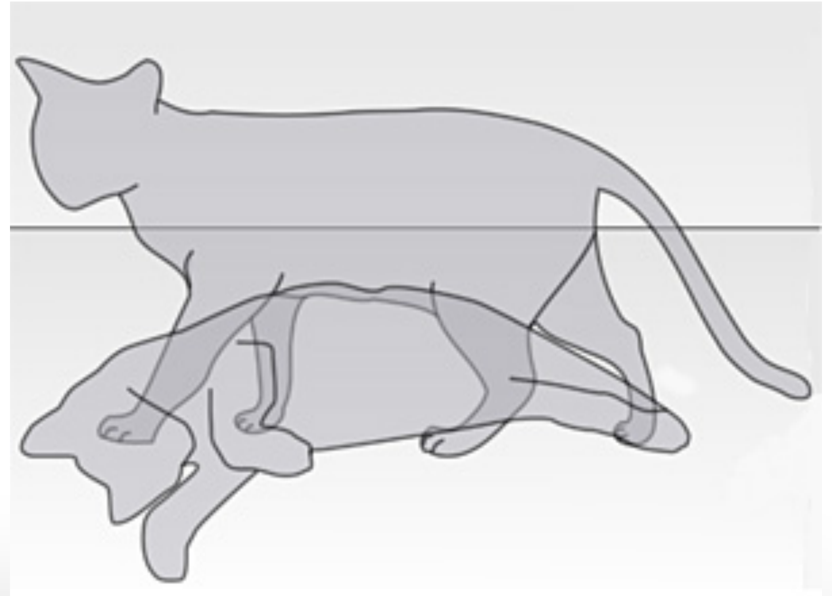
$$\langle \hat{A} \rangle = \langle \Psi(t) | \hat{A} | \Psi(t) \rangle.$$

$$\frac{d}{dt}\langle \hat{A} \rangle = \frac{1}{i\hbar}\langle \Psi(t) | \hat{A}\hat{H} - \hat{H}\hat{A} | \Psi(t) \rangle + \langle \Psi(t) | \frac{\partial A}{\partial t} | \Psi(t) \rangle$$

$$\boxed{\frac{d}{dt}\langle \hat{A} \rangle = \frac{1}{i\hbar}\langle [\hat{A}, \hat{H}] \rangle + \langle \frac{\partial \hat{A}}{\partial t} \rangle.}$$

$$\boxed{\text{If} \quad [\hat{H}, \hat{A}] = 0 \quad \text{and} \quad \frac{\partial \hat{A}}{\partial t} = 0 \implies \frac{d\langle \hat{A} \rangle}{dt} = 0 \implies \langle \hat{A} \rangle = \text{constant.}}$$

$$\boxed{\frac{1}{i\hbar}[\hat{A}, \hat{B}] \longrightarrow \{A, B\}_{classical}.}$$



Curiosity Kills the Cat

Lecture 06

Concluded