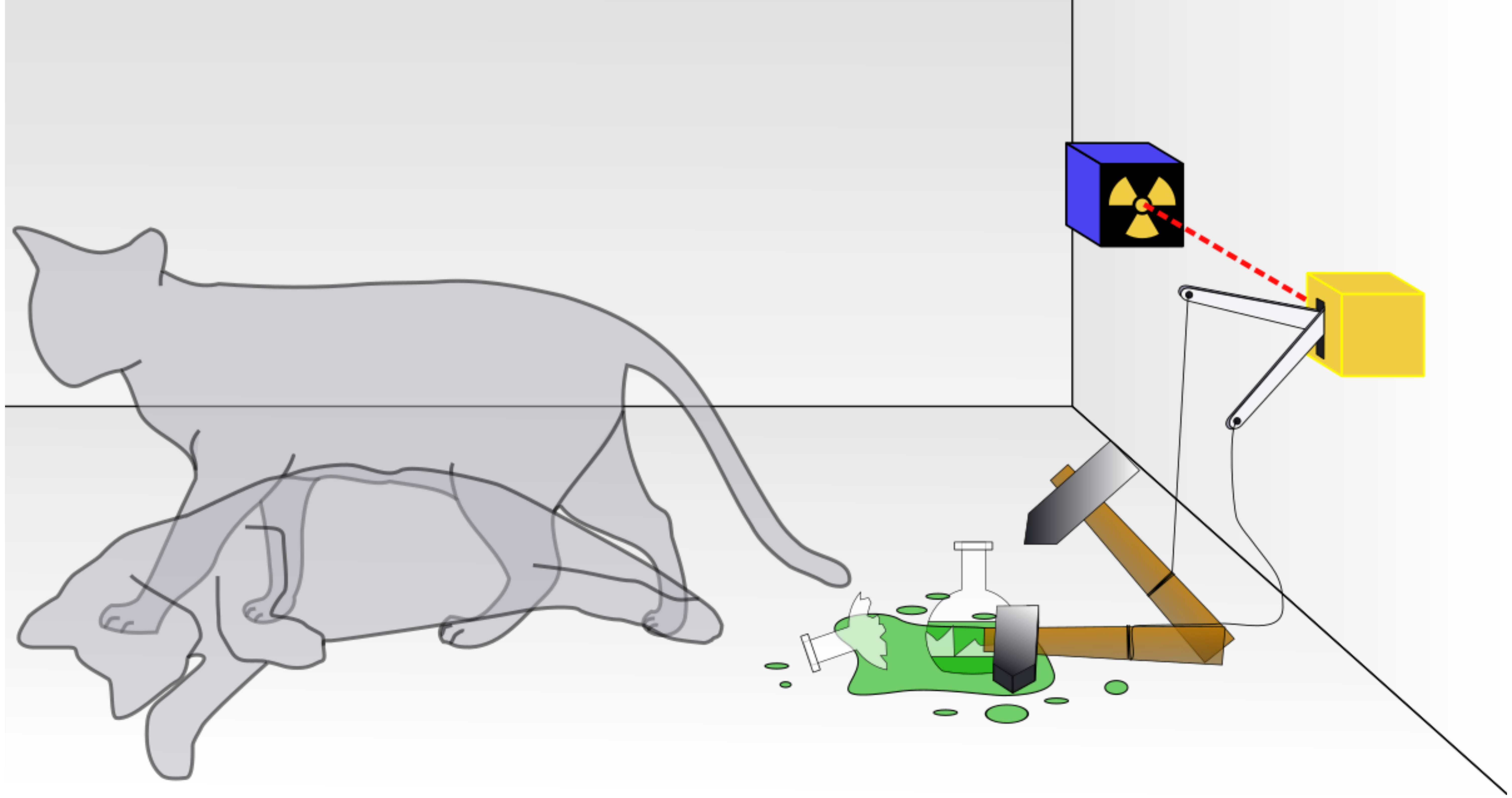
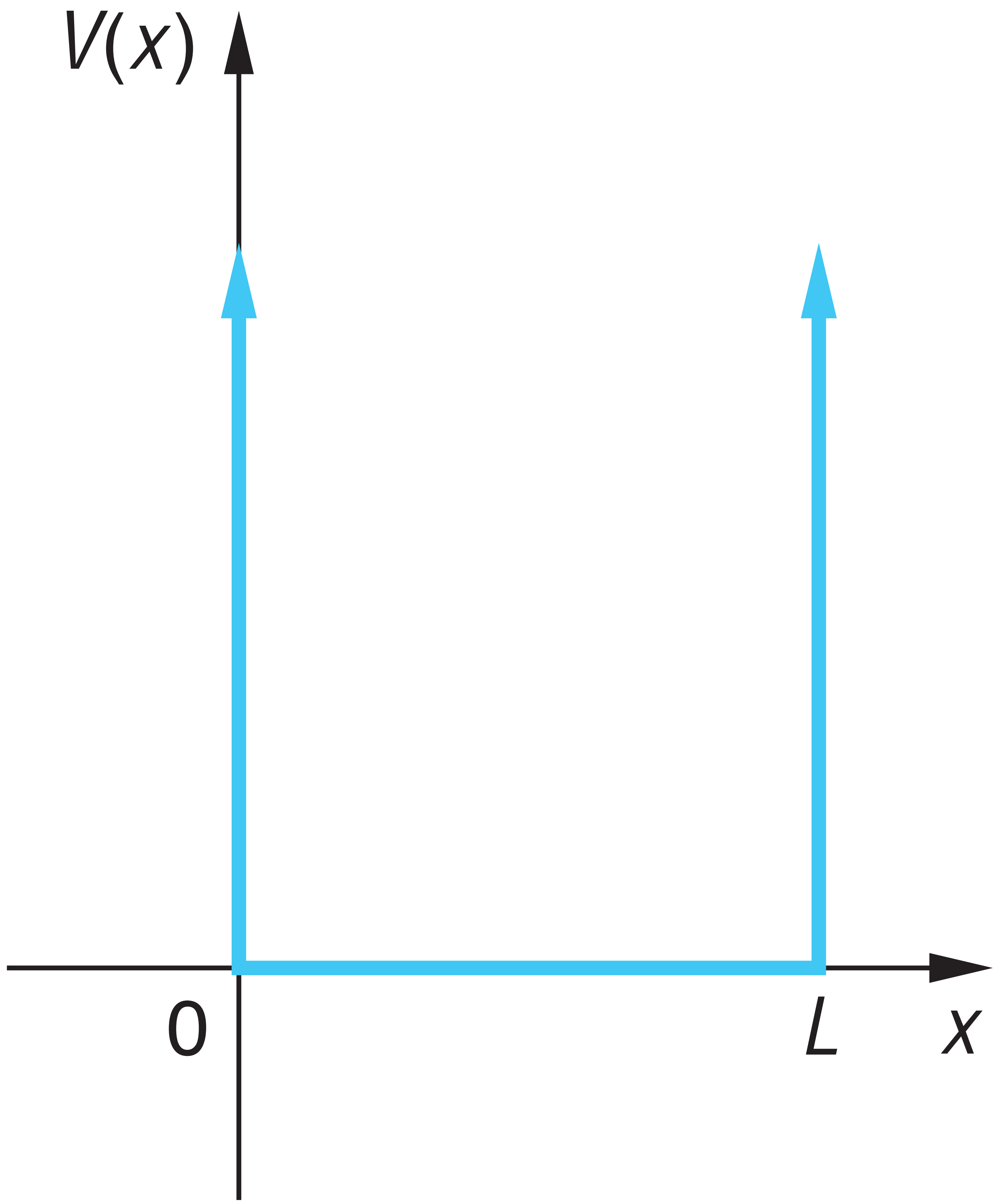
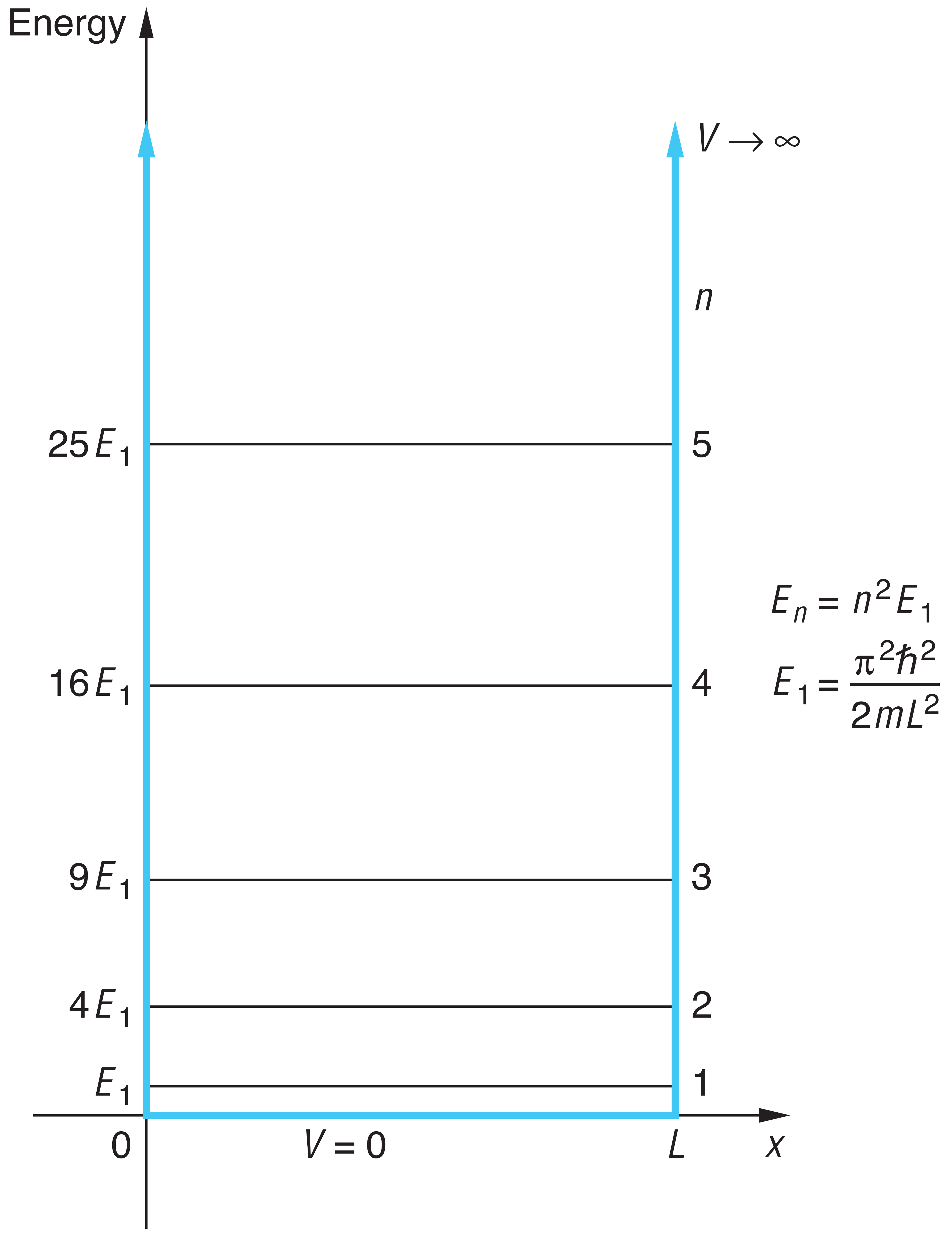


Lecture 05

Step Potential, Operators & expectation values

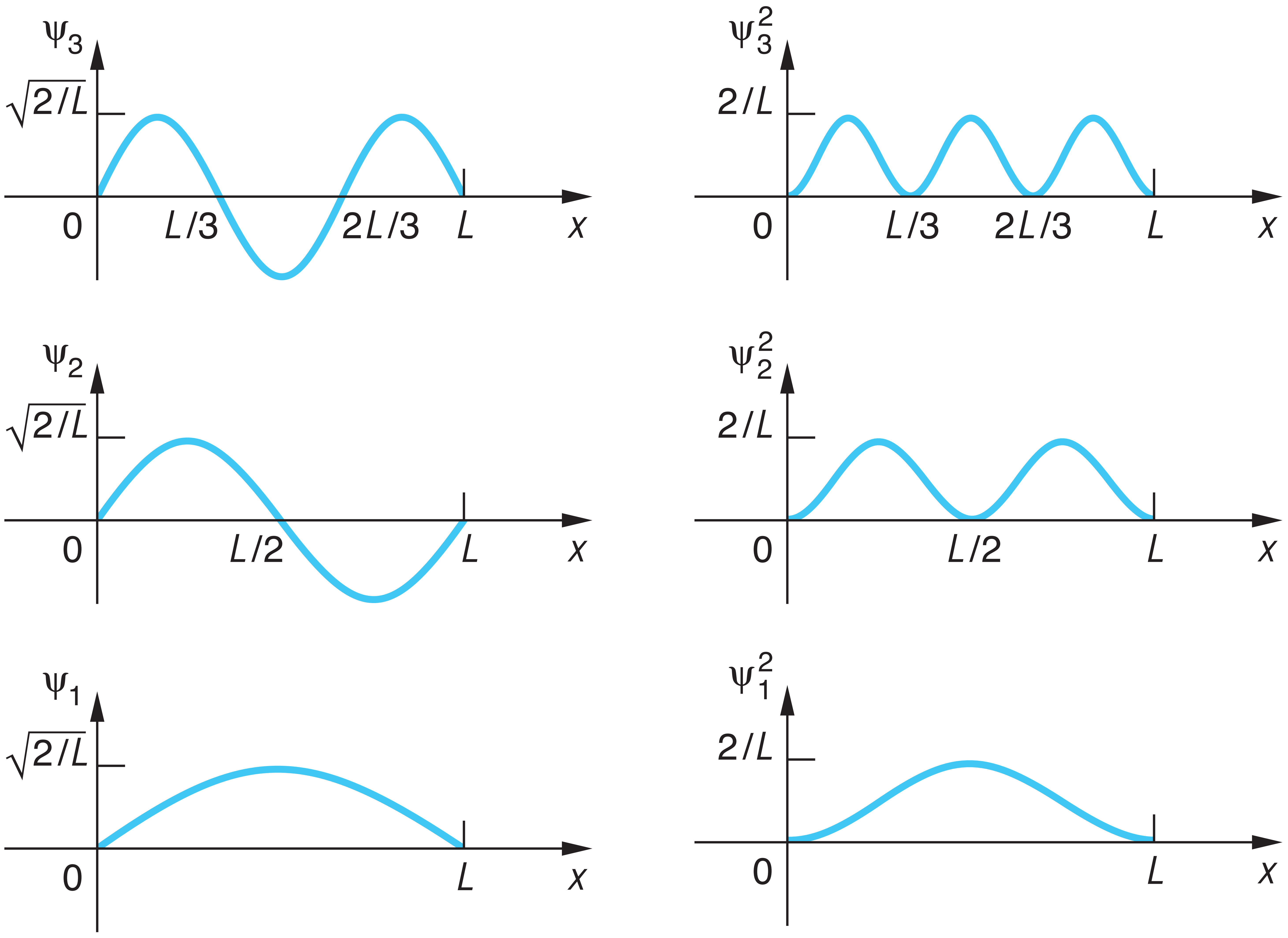
Infinite potential well





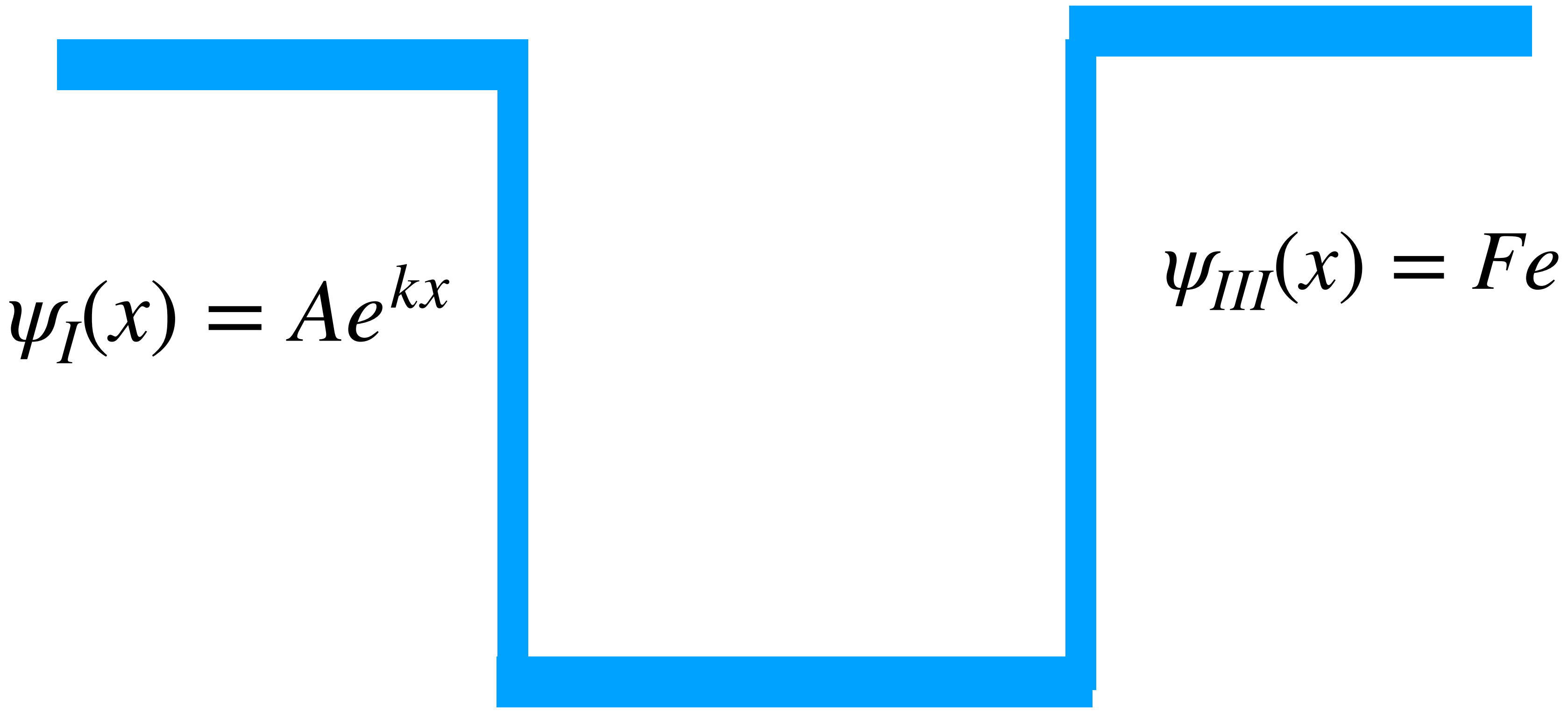
Infinite potential well

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \qquad n = 1, 2, 3, \ldots$$



Finite Potential Well

$$\psi_{II}(x) = Csink'x + Dcosk'x$$



$$\psi_I(x) = Ae^{kx}$$

$$\psi_{III}(x) = Fe^{-kx}$$

$$k^2 = -\frac{2mE}{\hbar^2} \qquad k'^2 = \frac{2m}{\hbar^2}(V_0 + E)$$

$$k^2 + k'^2 = \frac{2mV_0}{\hbar^2}$$

$$k'L = z \qquad z_0 = \frac{L}{\hbar}\sqrt{2mV_0}$$

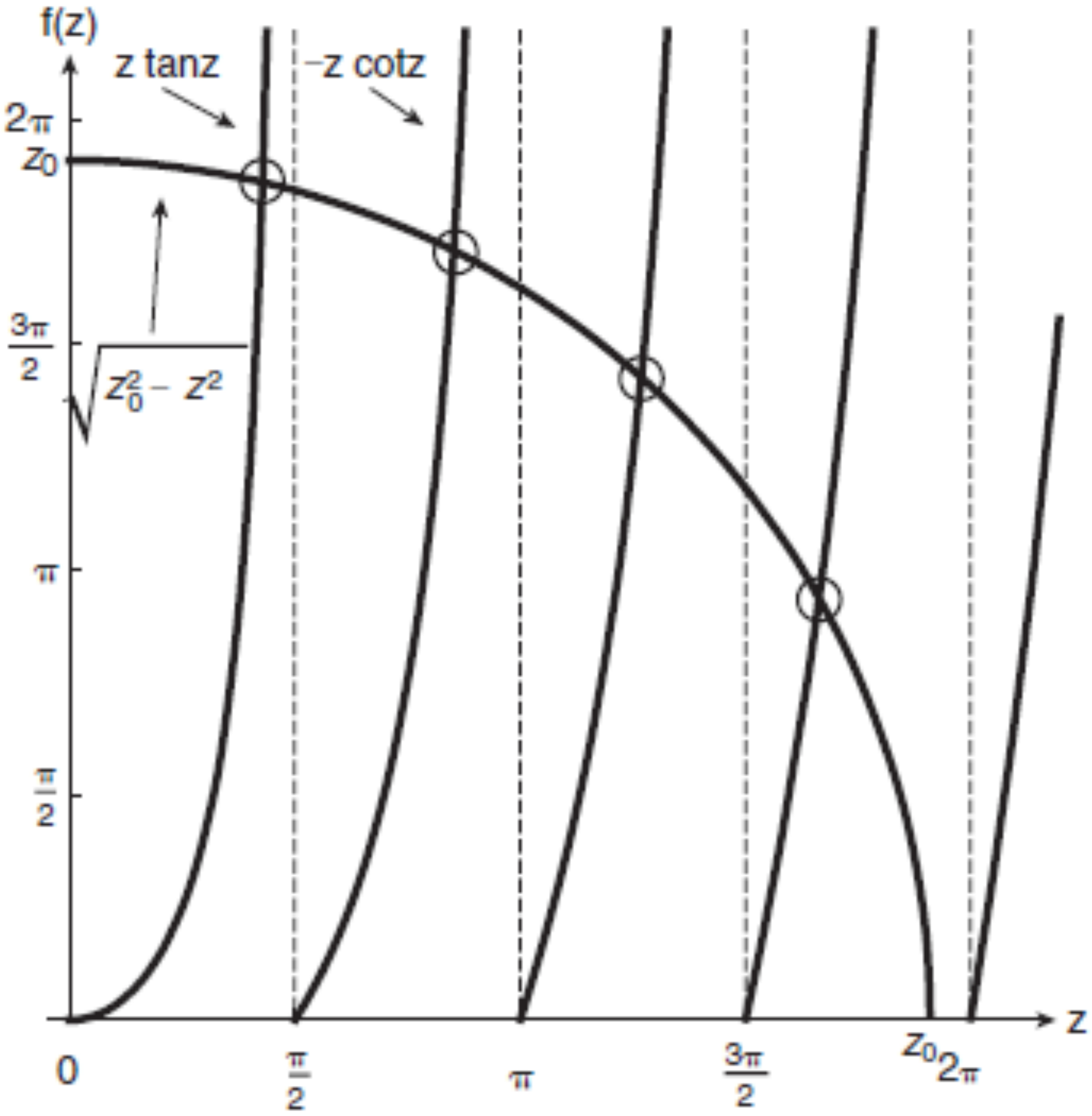
$$z^2 + (kL)^2 = z_0^2$$

$$kL = \sqrt{z_0^2 - z^2} \qquad k = k'tank'L$$

$$kL = k'Ltank'L$$

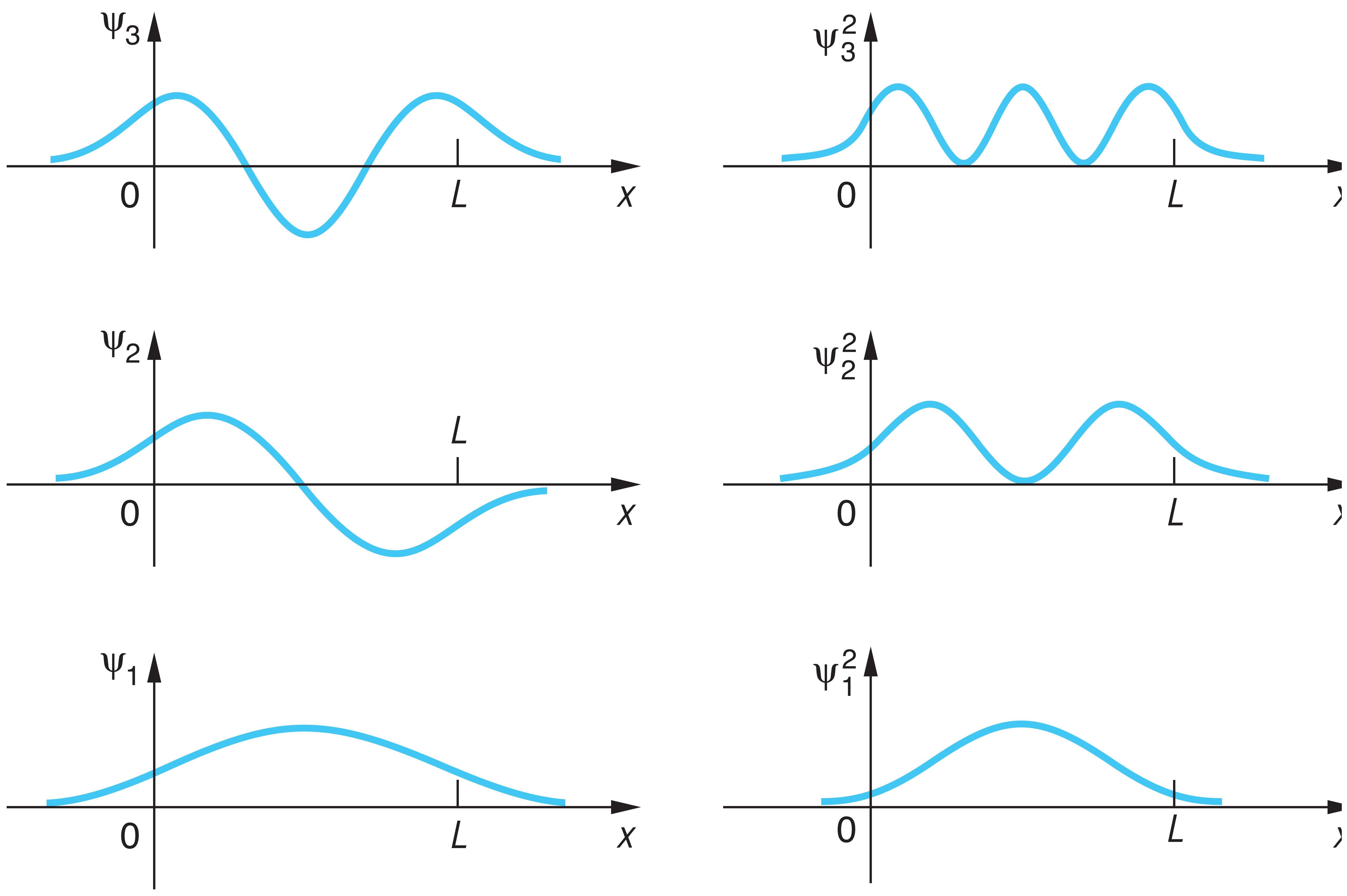
$$ztanz = \sqrt{z_0^2 - z^2}$$

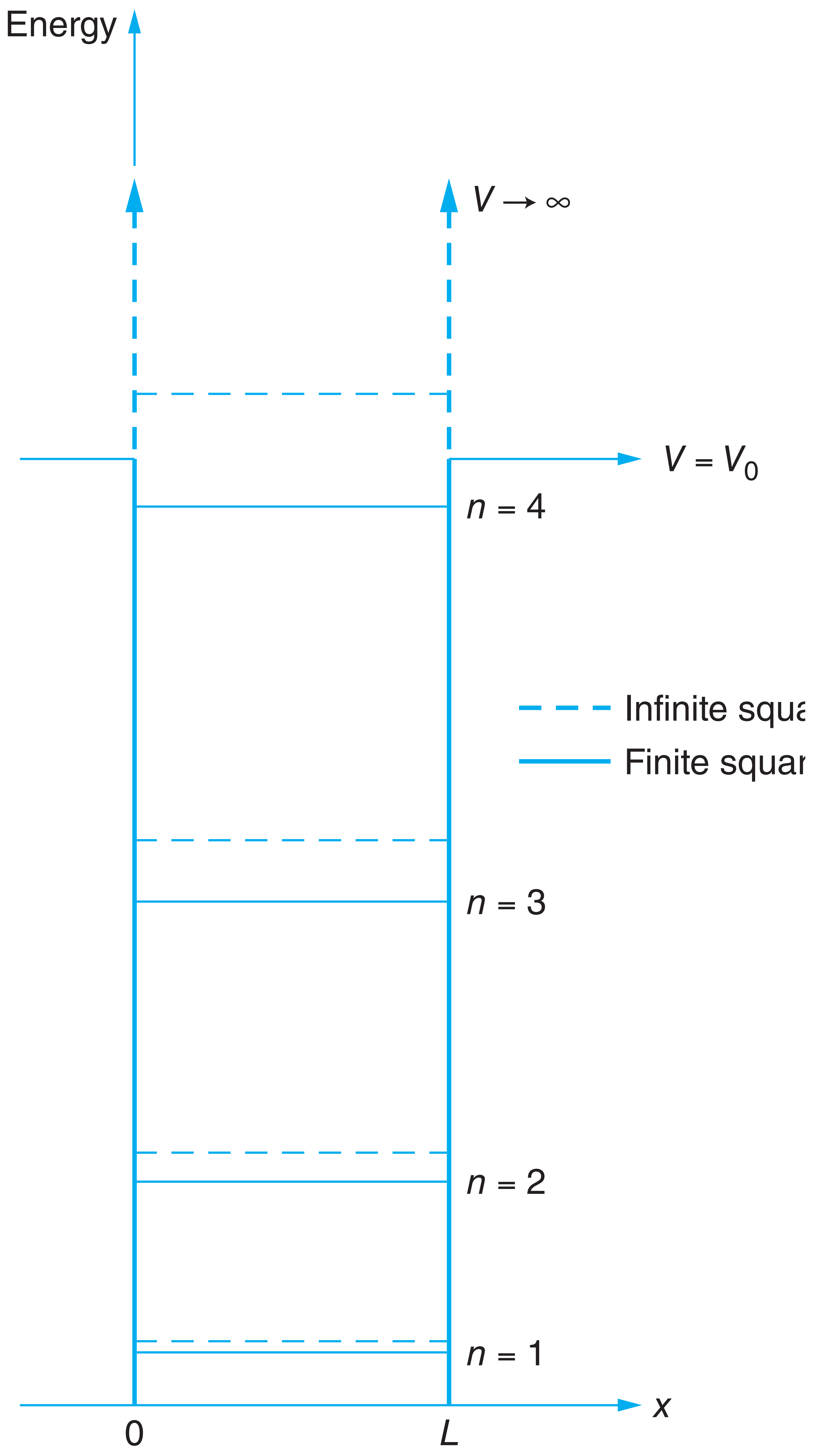
Finite Potential Well



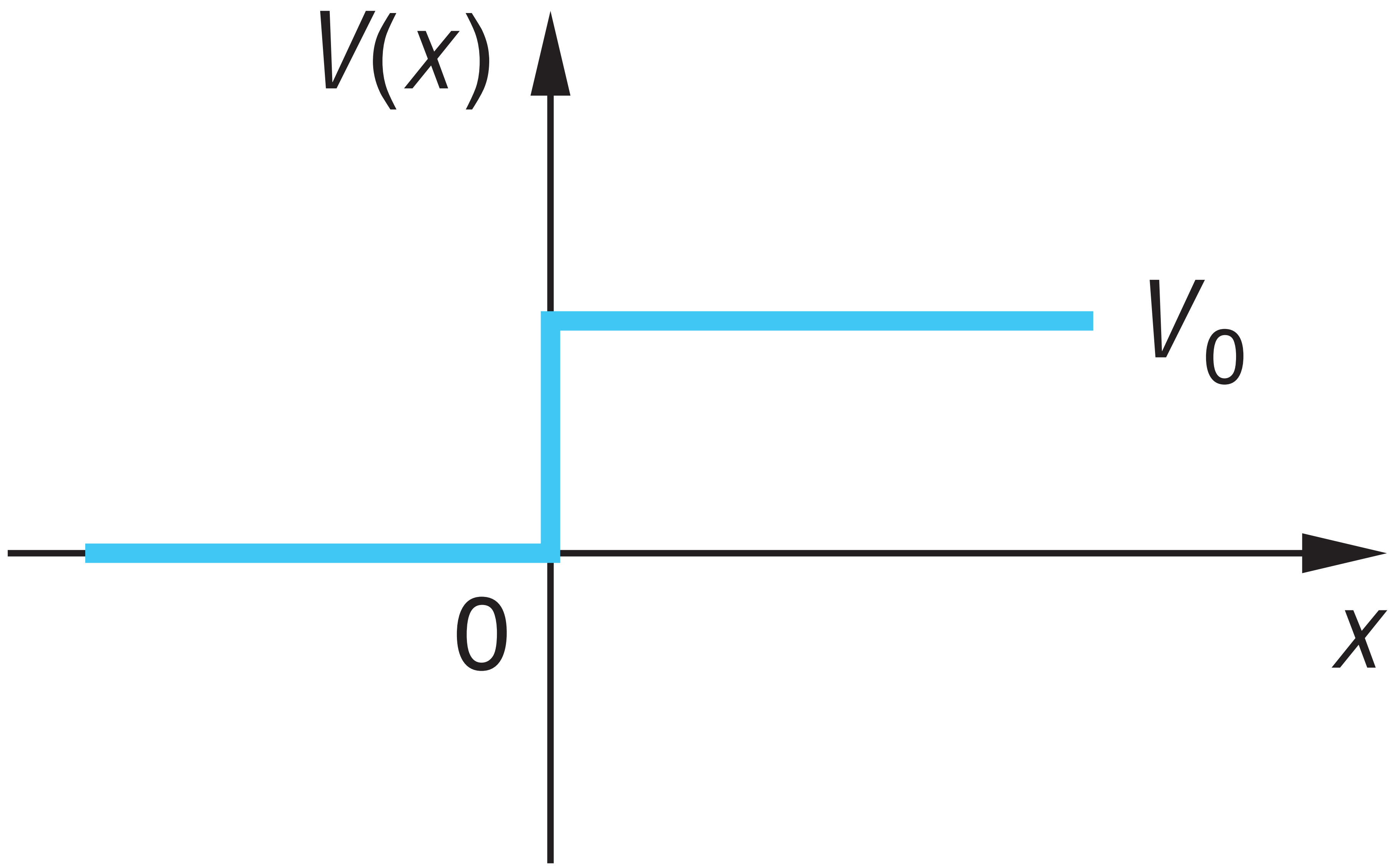
Graphical solution of the transcendental equations for the allowed energies of a finite square well ($z_0 = 6$).

Finite Potential Well





Step Potential



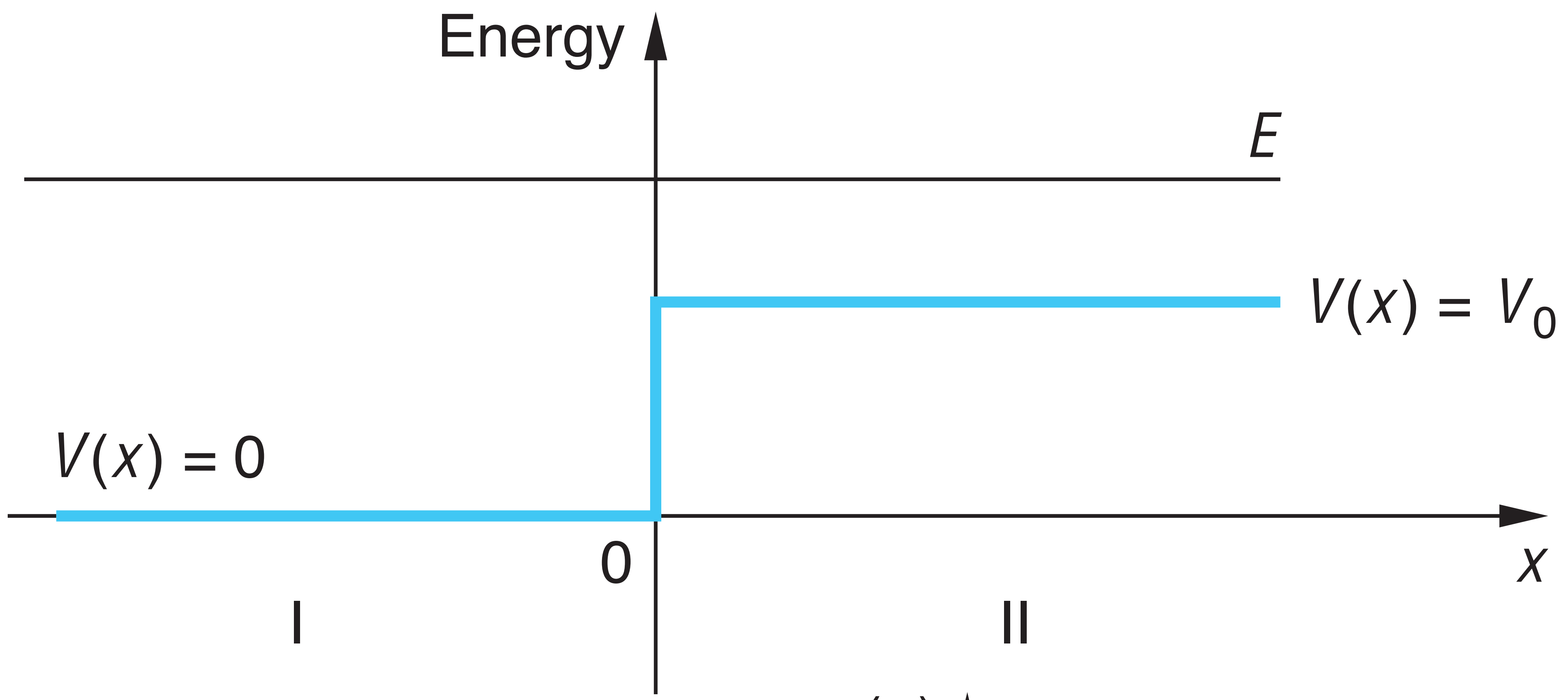
$$V(x) = 0 \quad \text{for} \quad x < 0$$
$$V(x) = V_0 \quad \text{for} \quad x > 0$$

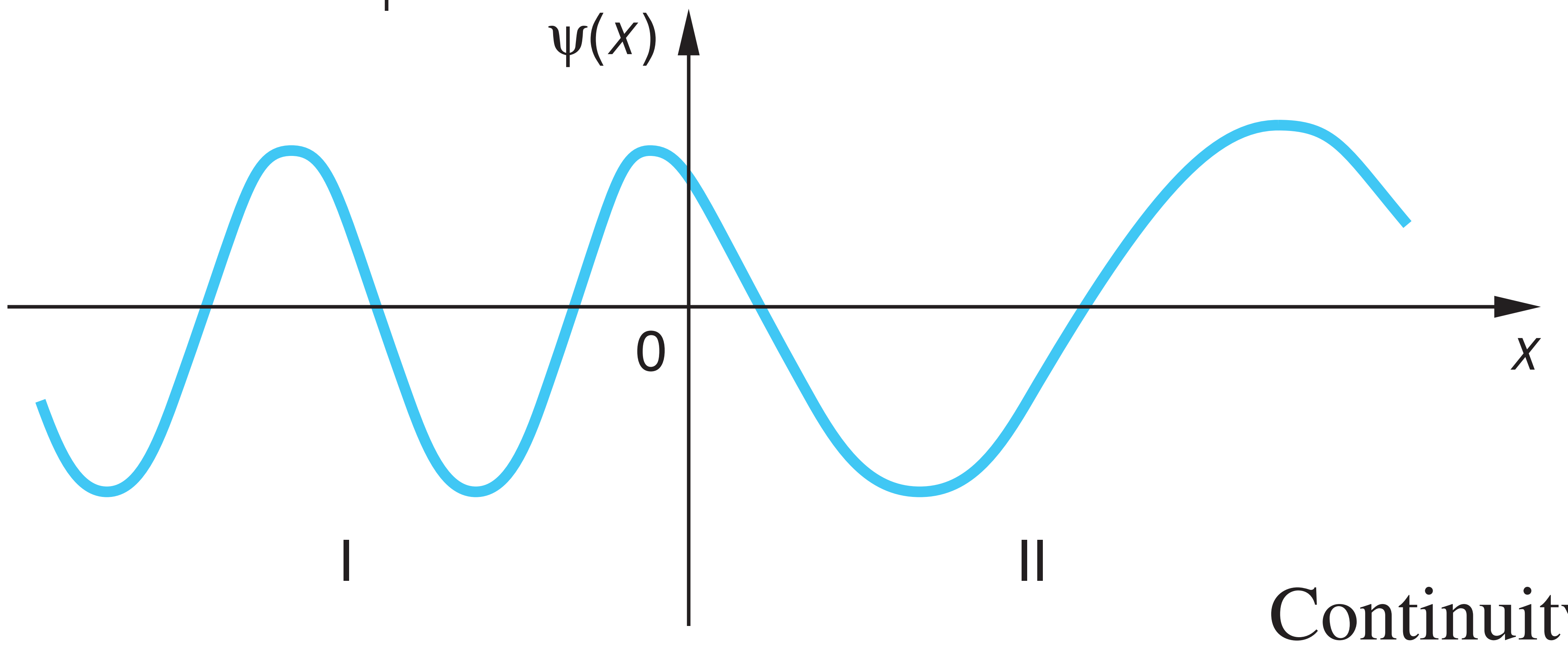
$$(x < 0) \qquad \frac{d^2\psi(x)}{dx^2} = -k_1^2\psi(x)$$

$$(x > 0) \qquad \frac{d^2\psi(x)}{dx^2} = -k_2^2\psi(x)$$

$$k_1 = \frac{\sqrt{2mE}}{\hbar} \quad \text{and} \quad k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

Step Potential





$$(x < 0) \qquad \psi_{\mathrm{I}}(x) = Ae^{ik_1 x} + Be^{-ik_1 x}$$

$$(x > 0) \qquad \psi_{\mathrm{II}}(x) = Ce^{ik_2 x} + De^{-ik_2 x}$$

$\psi_{\mathrm{I}}(0) = \psi_{\mathrm{II}}(0)$ and $d\psi(0)/dx = d\psi_{\mathrm{II}}(0)/dx$.

$$\psi_{\mathrm{I}}(0) = A + B = \psi_{\mathrm{II}}(0) = C$$

$$A + B = C$$

Continuity of $d\psi/dx$ at $x = 0$ gives

$$k_1 A - k_1 B = k_2 C$$

Step Potential

$$B = \frac{k_1 - k_2}{k_1 + k_2} A = \frac{E^{1/2} - (E - V_0)^{1/2}}{E^{1/2} + (E - V_0)^{1/2}} A$$

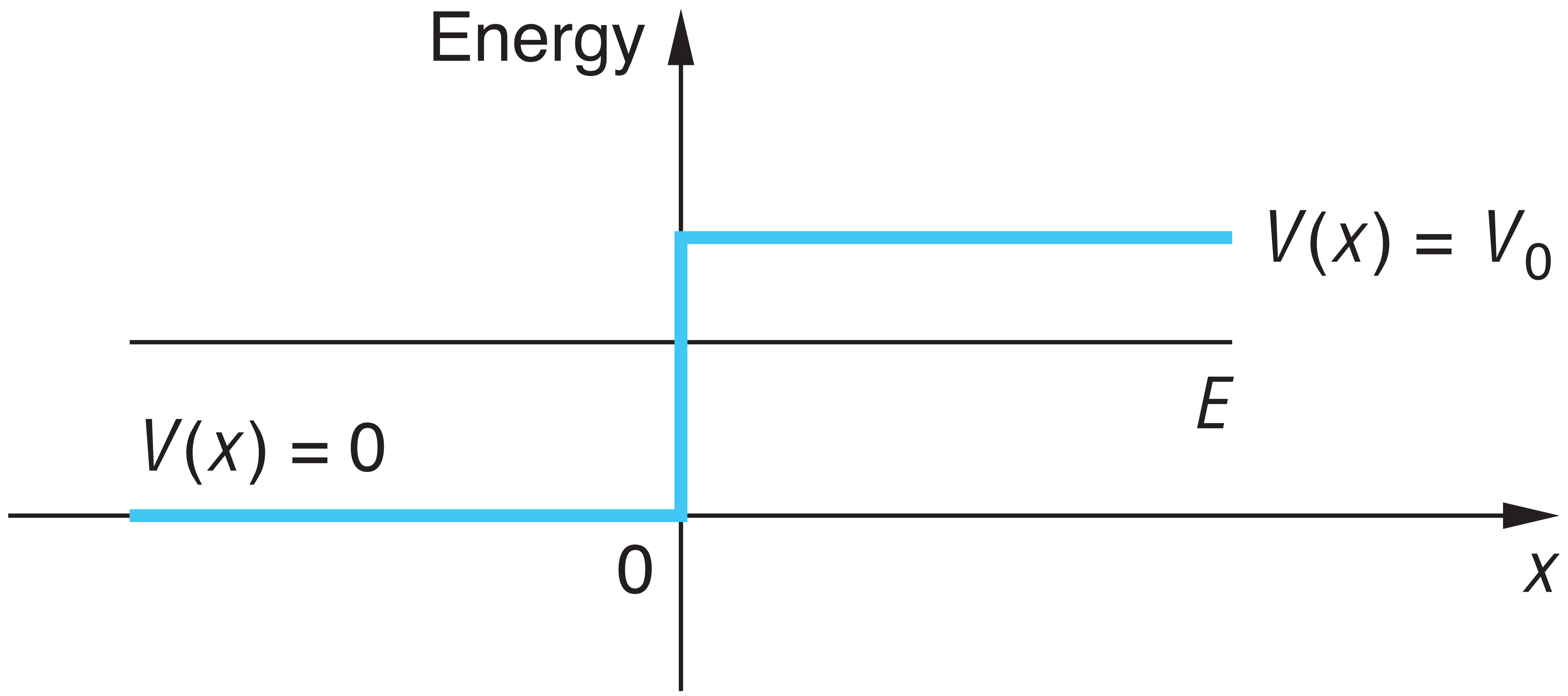
$$C = \frac{2k_1}{k_1 + k_2} A = \frac{2E^{1/2}}{E^{1/2} + (E - V_0)^{1/2}} A$$

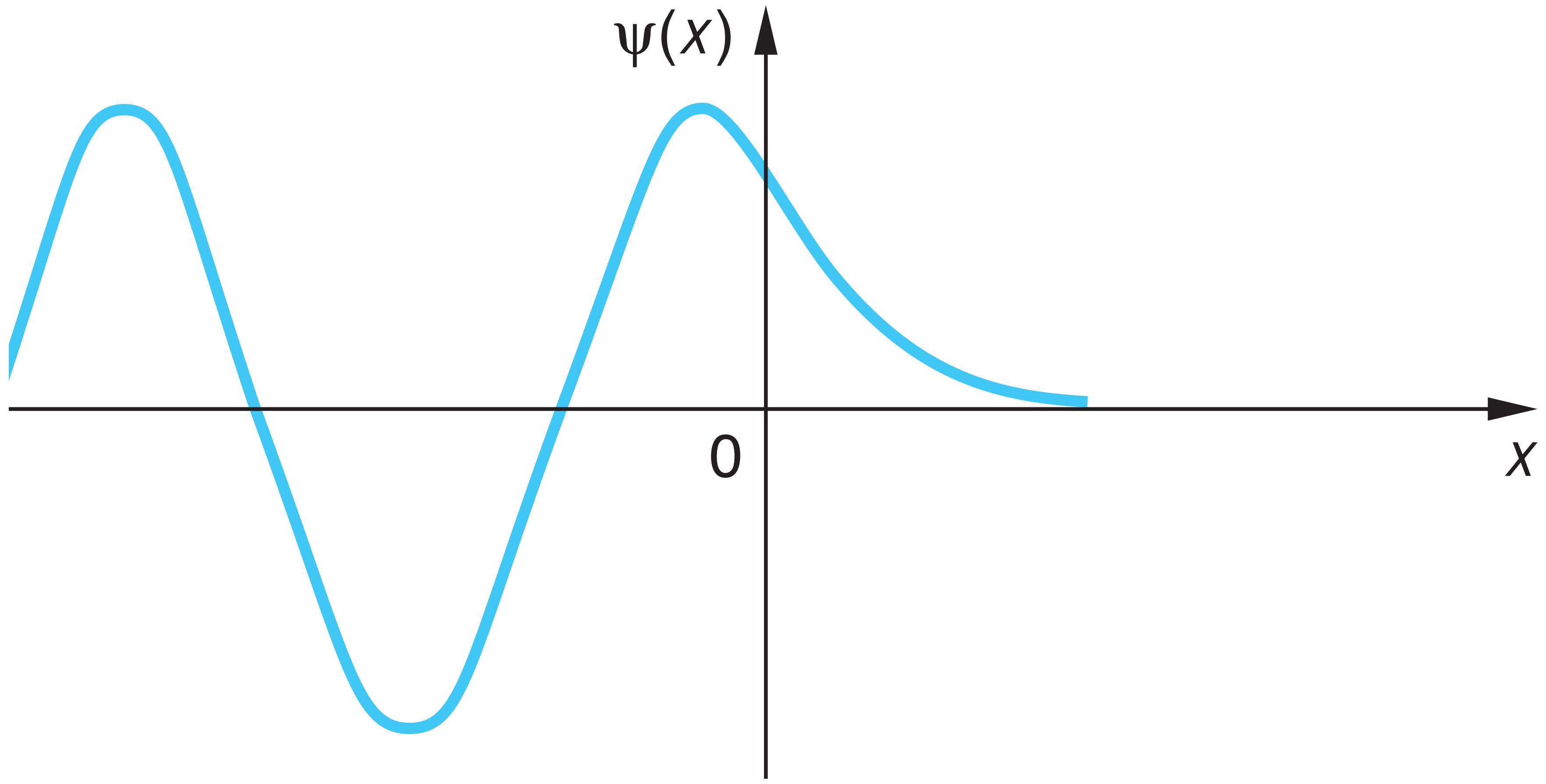
$$T + R = 1$$

$$R = \frac{|B|^2}{|A|^2} = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2$$

$$T = \frac{k_2}{k_1} \frac{|C|^2}{|A|^2} = \frac{4k_1 k_2}{(k_1 + k_2)^2}$$

Step Potential





$$E < V_0.$$

$$\psi_{II}(x) = Ce^{ik_2 x} = Ce^{-\alpha x}$$

$$\alpha = \sqrt{2m(V_0 - E)}/\hbar$$

$$\psi_{II} \rightarrow 0 \text{ as } x \rightarrow \infty$$

$$|B|^2 = |\mathbf{A}|^2, R = 1, \text{ and } T = 0.$$

$$|\psi_{II}|^2 = |C|^2 e^{-2\alpha x}$$

Operators and expectation values

$$dx/dt = \partial H/\partial p \text{ and } dp/dt = -\partial H/\partial x,$$

To every physically measurable quantity A, called an observable or dynamical variable, there corresponds a linear Hermitian operator $\hat{A}$ whose eigenvectors form a complete basis.

$$\hat{A}|\psi(t)\rangle = a_n|\psi_n\rangle,$$

$$f(\vec{r}, \vec{p}) \quad \longrightarrow \quad F(\hat{\vec{R}}, \hat{\vec{P}}) = f(\hat{\vec{R}}, -i\hbar\vec{\nabla}),$$

$$H = \frac{1}{2m}\vec{p}^{\,2} + V(\vec{r}, t)$$

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\hat{\vec{R}}, t),$$

Operators and expectation values

Symbol	Physical quantity	Operator
$f(x)$	Any function of x—the position x, the potential energy $V(x)$, etc.	$f(x)$
p_x	x component of momentum	$\frac{\hbar}{i}\frac{\partial}{\partial x}$
p_y	y component of momentum	$\frac{\hbar}{i}\frac{\partial}{\partial y}$
p_z	z component of momentum	$\frac{\hbar}{i}\frac{\partial}{\partial z}$

Operators and expectation values

Symbol	Physical quantity	Operator
E	Hamiltonian (time independent)	$\frac{p_{\text{op}}^2}{2m} + V(x)$
E	Hamiltonian (time dependent)	$i\hbar \frac{\partial}{\partial t}$
E_k	Kinetic energy	$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$
L_z	z component of angular momentum	$-i\hbar \frac{\partial}{\partial \phi}$

Operators and expectation values

$$\langle x \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t)\, x \Psi(x,t)\, dx$$

$$\langle x \rangle = \int_{-\infty}^{+\infty} \psi^*(x)\, x \psi(x)\, dx$$

$$\langle p \rangle = \int_{-\infty}^{+\infty} \Psi^* \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi\, dx$$

$$\langle p^2 \rangle = \int_{-\infty}^{+\infty} \Psi^* \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right) \Psi\, dx$$

Expectation values of p and p^2 for ground state of Infinite well

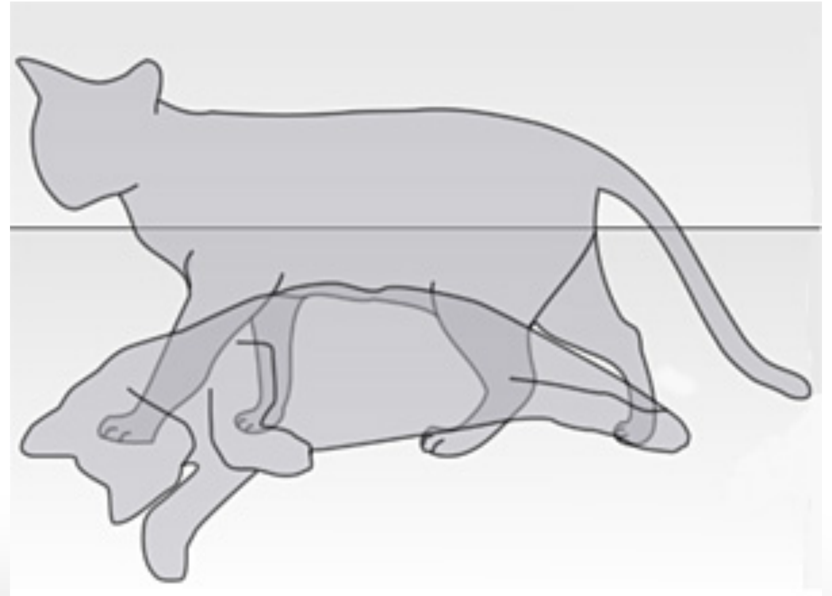
$$\langle p \rangle = \int_0^L \left(\sqrt{\frac{2}{L}}\sin\frac{nx}{L}\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\left(\sqrt{\frac{2}{L}}\sin\frac{nx}{L}\right)dx$$

$$= \frac{\hbar}{i}\frac{2}{L}\frac{\pi}{L}\int_0^L \sin\frac{\pi x}{L}\cos\frac{\pi x}{L}dx = 0$$

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi = -\hbar^2\frac{\partial^2\psi}{\partial x^2} = -\hbar^2\left(-\frac{\pi^2}{L^2}\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}\right)$$

$$= +\frac{\hbar^2\pi^2}{L^2}\psi$$

$$\langle p^2 \rangle = \frac{\hbar^2\pi^2}{L^2}\int_0^L \psi^*\psi\, dx = \frac{\hbar^2\pi^2}{L^2}$$



Curiosity Kills the Cat

Lecture 05

Concluded