Unit 1

1. Index Notation

- It is the notation for vector, tensor equations. For example: $\vec{F} = m\vec{a}$
- In cartesian co-ordinates $F_1 = ma_1, F_2 = ma_2, F_3 = ma_3$
- In index notation $F_i = ma_i$ for i=1,2,3
- Similarly, $a_i + \alpha b_i = c_i$ is the index notation for $\vec{a} + \alpha\vec{b} = \vec{c}$ for i=1,2,3

Rule 1: The range of indices is removed. It is assumed that the range is known from the context.

$a_i + \alpha b_j = c_i$ for i=1,2,3; for j=1,2,3. The notation is not valid
Reason 1: Different free indices, i and j
Reason 2: Range of indices is defined

Corollary to rule 1: The free index in each term of the equation has to be the same.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}, C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ and $C = A + 3B$

Index notation

$C_{ij} = A_{ij} + 3B_{ij} \rightarrow Valid$ i and j are free indices

$$C_{ik} = A_{ij} + 3B_{ij} \rightarrow Invalid$$

For

$$C = A + 3B^T$$

$$C_{ij} = A_{ij} + 3B_{ji} \rightarrow Valid$$

Rule 2: Einstein summation Convention

- If an index repeats (dummy index), it means summation over that index
- For example:

$$\vec{a}.\vec{b} = \gamma$$

$$\Rightarrow a_1b_1 + a_2b_2 + a_3b_3 = \gamma$$

$$\Rightarrow \gamma = \sum_{i=1}^{3} a_ib_i = \sum a_ib_i \rightarrow a_ib_i$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

- Matrix notation: $Ay = B$
- Index Notation

$a_{11}y_1 + a_{12}y_2 = B_1 \Longrightarrow \sum_{j=1}^{2} a_{1j}\, y_j = B_1$ (1)

$a_{21}y_1 + a_{22}y_2 = B_2 \Longrightarrow \sum_{j=1}^{2} a_{2j}\, y_j = B_2$ (2)

$$\Rightarrow \sum_{i,j=1}^{2} a_{ij}y_j = B_i \ or\ a_{ij}y_j = B_i$$

- Notice:

Free index, i. Is same on both sides

j is a dummy index

Corollary to rule 2

In a single term, an index should not be repeated more than two times.

For example

$$(\vec{a}.\vec{b})\vec{c} - (\vec{c}.\vec{a})\vec{b}$$

$$(a_i.b_i)c_j - (c_i.a_i)b_j$$

$$(a_i.b_i)c_i \rightarrow not\ valid$$

Kronecker Delta Function, δ_{ij}

$$\delta_{ij} = 1\ if\ i = j$$

$$\delta_{ij} = 0\ if\ i \neq j$$

In a 2D matrix

$$\delta = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

An important property of Kronecker Delta Function

$$\delta_{ij}u_j = u_i$$

TASK: Prove it

The Alternating Tensor, ε_{ijk}

$$\varepsilon_{ijk} = 1\ if\ ijk = 123{,}231\ or\ 312$$

$$\varepsilon_{ijk} = -1\ if\ ijk = 213, 132\ or\ 321$$

$$\varepsilon_{ijk} = 0\ otherwise$$

It can be used to define cross product

If $\vec{c} = \vec{a} \times \vec{b}, then\ c_i = \varepsilon_{ijk}a_jb_k$

Also,

$$\varepsilon_{ijk}\varepsilon_{lmk} = \varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

2. **Transformation of axes**

   If xyz is the orthogonal coordinate system in which stress, σ is represented and $x'y'z'$ is the new axis in which σ is to be represented, then the general form of the transformation is:

$$\sigma_{xx'} = l_{x'x}l_{x'x}\sigma_{xx} + l_{x'y}l_{x'x}\sigma_{yx} + l_{x'z}l_{x'x}\sigma_{zx} + l_{x'x}l_{x'y}\sigma_{xy} + l_{x'y}l_{x'y}\sigma_{yy} + l_{x'z}l_{x'y}\sigma_{zy} + l_{x'x}l_{x'z}\sigma_{xz} + l_{x'y}l_{x'z}\sigma_{yz} + l_{x'z}l_{x'z}\sigma_{zz}$$

   And

$$\sigma_{x'y'} = l_{x'x}l_{y'x}\sigma_{xx} + l_{x'y}l_{y'x}\sigma_{yx} + l_{x'z}l_{y'x}\sigma_{zx} + l_{x'x}l_{y'y}\sigma_{xy} + l_{x'y}l_{y'y}\sigma_{yy} + l_{x'z}l_{y'y}\sigma_{zy} + l_{x'x}l_{y'z}\sigma_{xz} + l_{x'y}l_{y'z}\sigma_{yz} + l_{x'z}l_{y'z}\sigma_{zz}$$

   Where $l_{x'x}$ is the cosine of the angle between x' and x and hence the rest.

3. **Stress**
   - Stress, σ, is defined as the intensity of force at a point
     $$\sigma = {}^{\partial F}/_{\partial A} \; as \; \partial A \to 0$$
   - If the state of stress is the same everywhere in a body, $\sigma = {}^{F}/_{A}$
   - A normal stress (compressive or tensile) is the one in which force acts on the area that is normal to it whereas in shear force, force is parallel to the area.
   - σ_{xx} indicates that the force is in x-direction and it acts on a plane normal to x
   - σ_{xx} indicates that the force is in y-direction and is acting on a plane normal to x
   - In tensor notation, the state of stress is expressed as
     $$\sigma_{ij} = \begin{bmatrix} \sigma_{xx} & \sigma_{yx} & \sigma_{zx} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{zy} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix}$$
     $$\sigma_{ij} = \frac{F_i}{A_j}$$

$$\sigma_{xx} = \sigma_x, \sigma_{xy} = \tau_{xy}$$

$$\sigma_{ij} = \sigma_{ji} \; or \; \tau_{ij} = \tau_{ji}$$

<u>Principal Stresses</u>

In a set of axis where no shear stresses exist and only normal stresses are present, these normal stresses, $\sigma_{1,}\sigma_2 and\ \sigma_3$ are called principal stresses and 1,2 and 3 axes are called the principal stress axes. The magnitude of the principal stresses, σ_p, are the roots of the equation

$$\sigma_p^3 - I_1\sigma_p^2 - I_2\sigma_p - I_3 = 0$$

Where

$$I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \frac{-p}{3}, where\ p\ is\ pressure$$

$$I_2 = \sigma_{yz}^2 + \sigma_{zx}^2 + \sigma_{xy}^2 - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx} - \sigma_{xx}\sigma_{yy}$$

$$I_3 = \sigma_{xx}\sigma_{yy}\sigma_{zz} + 2\sigma_{yz}\sigma_{zx}\sigma_{xy} - \sigma_{xx}\sigma_{yz}^2 - \sigma_{yy}\sigma_{zx}^2 - \sigma_{zz}\sigma_{xy}^2$$

$I_1, I_2\ and\ \ I_3$ are called stress invariants

In terms of principal stresses, the invariants are

$$I_1 = \sigma_1 + \ \sigma_2 + \ \sigma_3$$
$$I_3 = \ -\sigma_{22}\sigma_{33} - \sigma_{33}\sigma_{11} - \sigma_{11}\sigma_{22}$$
$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33}$$

<u>Plane Stress</u>

- A stress condition in which the stresses in one of the primary directions is zero

<u>Deviatoric Stress or stress deviator, σ'</u>
Component of the total stress that causes plastic deformation

<u>Hydrostatic stress, σ''</u>
The other component of stress (apart from stress deviator) is called spherical or hydrostatic stress

$$\sigma'' = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = -p$$

The stress devaitor is given by

$$\sigma_1' = \sigma_1 - \sigma_1'' = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3}$$
$$\sigma_2' = \sigma_2 - \sigma_2'' = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{3}$$
$$\sigma_3' = \sigma_3 - \sigma_3'' = \frac{2\sigma_3 - \sigma_2 - \sigma_1}{3}$$

The Principal Stress Deviators are the roots of the cubic equation

$$(\sigma')^3 - J_2\sigma' - J_3 = 0$$

$$J_2 = \frac{-1}{6}[(\sigma_1 - \sigma_2)^2 - (\sigma_2 - \sigma_3)^2 - (\sigma_3 - \sigma_1)^2]$$

$$J_3 = \frac{1}{27}[(2\sigma_1 - \sigma_2 - \sigma_3) - (2\sigma_2 - \sigma_3 - \sigma_1) - (2\sigma_3 - \sigma_1 - \sigma_2)]$$

4. Strain

- Linear strain is defined as the ratio of change in length to the original length of the same dimension
- Engineering or normal Strain, $e = \frac{\delta}{L_o} = \frac{\Delta L}{L_o} = \frac{L - L_o}{L_o}$
- True strain, $\varepsilon = \ln\frac{L}{L_o}$, where L is the final length

$$\varepsilon = \ln\frac{L}{L_o} = ln(1+e)$$

- True Stress, $\sigma = s(1+e)$

- The transformation of axes for strain are same as that for stress (refer to transformation of stress)

Task: Using the property for shear stresses and strains ($\sigma_{12} = \sigma_{21}\ and\ same\ for\ strain$), rewrite their transformation of axes equation.

<u>Plane Strain</u>

- Strain condition where all the deformation is confined to xy or yz or xz plane

- Volumetric Strain, $\Delta = e_1 + e_2 + e_3$

- In an analogous manner, the strain at any point can be divided into deviator and hydrostatic strain

$$e_1' = e_1 - e_1'' = \frac{2e_1 - e_2 - e_3}{3}$$
$$e_2' = e_2 - e_2'' = \frac{2e_2 - e_1 - e_3}{3}$$
$$e_3' = e_3 - e_3'' = \frac{2e_3 - e_2 - e_1}{3}$$
$$e'' = \frac{e_1 + e_2 + e_3}{3}$$

5. Stress-strain relations

- $\sigma_x = Ee_x$ Hooke's Law
- Generalised Hooks law

$$e_1 = \frac{1}{E}[\sigma_1 - \nu(\sigma_2 + \sigma_3)]$$
$$e_2 = \frac{1}{E}[\sigma_2 - \nu(\sigma_3 + \sigma_1)]$$

$$e_3 = \frac{1}{E}[\sigma_3 - \nu(\sigma_2 + \sigma_1)]$$

$\nu = Poisson's\ ratio$
$= ratio\ of\ strain\ in\ the\ transverse\ direction\ andin\ longitudinal\ direction$