Module 3: Analysis of Strain

3.2.1 MOHR'S CIRCLE FOR STRAIN

The Mohr's circle for strain is drawn and that the construction technique does not differ from that of Mohr's circle for stress. In Mohr's circle for strain, the normal strains are plotted on the horizontal axis, positive to right. When the shear strain is positive, the point representing the x-axis strains is plotted at a distance $\frac{\gamma}{2}$ below the ε-line; and the y-axis point a distance $\frac{\gamma}{2}$ above the ε-line; and vice versa when the shear strain is negative.

By analogy with stress, the principal strain directions are found from the equations

$$\tan 2\theta = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \qquad (3.19)$$

Similarly, the magnitudes of the principal strains are

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad (3.20)$$

3.2.2 EQUATIONS OF COMPATABILITY FOR STRAIN

Expressions of compatibility have both mathematical and physical significance. From a mathematical point of view, they assert that the displacements u, v, w are single valued and continuous functions. Physically, this means that the body must be pieced together.

The kinematic relations given by Equation (3.3) connect six components of strain to only three components of displacement. One cannot therefore arbitrarily specify all of the strains as functions of x, y, z. As the strains are not independent of one another, in what way they are related? In two dimensional strain, differentiation of ε_x twice with respect to y, ε_y twice with respect to x, and γ_{xy} with respect to x and y results in

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}, \quad \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y}$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} \qquad (3.21)$$

or $\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$

This is the condition of compatibility of the two dimensional problem, expressed in terms of strain. The three-dimensional equations of compatibility are derived in a similar manner:

Thus, in order to ensure a single-valued, continuous solution for the displacement components, certain restrictions have to be imposed on the strain components. These resulting equations are termed the compatibility equations.

Suppose if we consider a triangle ABC before straining a body [Figure 3.4(a)] then the same triangle may take up one of the two possible positions Figure 3.4(b) and Figure 3.4(c)) after straining, if an arbitrary strain field is specified. A gap or an overlapping may occur, unless the specified strain field obeys the necessary compatibility conditions.

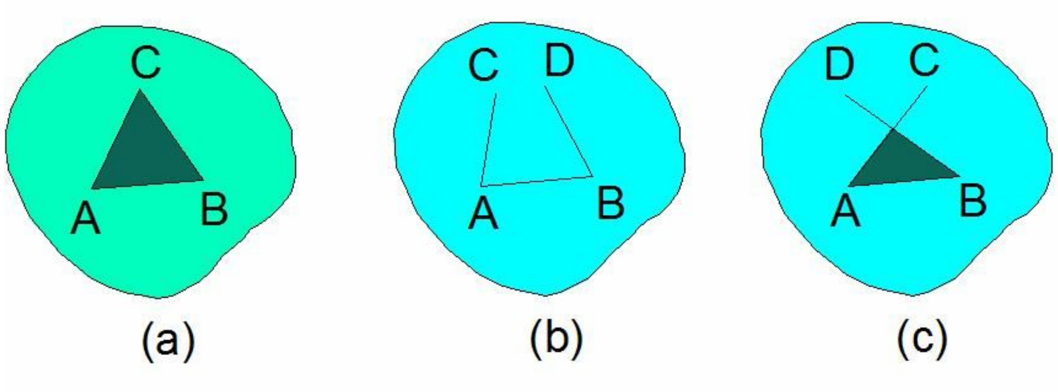

Fig. 3.4 Strain in a body

Now,

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad (3.23)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad (3.23a)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \quad (3.23b)$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad (3.23c)$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \quad (3.23d)$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{3.23e}$$

Differentiating Equation (3.23) with respect to *y* and Equation (3.23a) with respect to *x* twice, we get

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} \tag{3.23f}$$

$$\frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial y \partial x^2} \tag{3.23g}$$

Adding Equations (3.23f) and (3.23g), we get

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial y \partial x^2} \tag{3.23h}$$

Taking the derivative of Equation (3.23c) with respect to *x* and *y* together, we get

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 v}{\partial y \partial x^2} + \frac{\partial^3 u}{\partial x \partial y^2} \tag{3.23i}$$

From equations (3.23h) and (3.23i), we get

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{3.23j}$$

Similarly, we can get

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \tag{3.23k}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial x \partial z} \tag{3.23l}$$

Now, take the mixed derivative of Equation (3.23) with respect to *z* and *y*,

$$\therefore \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z} \tag{3.23m}$$

And taking the partial derivative of Equation (3.23c) with respect to *z* and *x*, we get

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z} + \frac{\partial^3 v}{\partial z \partial x^2} \tag{3.23n}$$

Also taking the partial derivative of Equation (3.23d) with respect to *x* twice, we get

$$\frac{\partial^2 \gamma_{yz}}{\partial x^2} = \frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 v}{\partial x^2 \partial z} \tag{3.23p}$$

And take the derivative of Equation (3.23e) with respect to y and x

Thus, $$\frac{\partial^2 \gamma_{zx}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial z} + \frac{\partial^3 w}{\partial x^2 \partial y} \tag{3.23q}$$

Now, adding Equations (3.23n) and (3.23q) and subtracting Equation (3.23p), we get

$$-\left(\frac{\partial^2 \gamma_{yz}}{\partial x^2}\right) + \frac{\partial^2 \gamma_{xz}}{\partial x \partial y} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial z} = \frac{2\partial^3 u}{\partial x \partial y \partial z} \tag{3.23r}$$

By using Equation (3.23m), we get

$$\frac{2\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left[-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right] \tag{3.23s}$$

Similarly, we can get

$$\frac{2\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left[-\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yx}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right] \tag{3.23t}$$

$$\frac{2\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left[-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right] \tag{3.23u}$$

Thus the following are the six compatibility equations for a three dimensional system.

$$\begin{aligned}
&\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\
&\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\
&\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \\
&\frac{2\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\
&\frac{2\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\
&\frac{2\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)
\end{aligned} \tag{3.24}$$

3.2.3 MEASUREMENT OF SURFACE STRAINS - STRAIN ROSETTES

Strain Rosettes

Whenever a material is subjected to plane stress, it is desirable to obtain the stresses by direct measurement. As the stresses cannot be measured directly, it is essential to measure the strains or deformation that takes place in the material during loading. These strains or deformations are measured with sensitive strain gauges attached to the surface of the body before it is loaded so that these gauges can record the amount of strain that takes place during loading. It is more accurate and easier to measure in the neighbourhood of a chosen point on the surface of the body, the linear strains in different directions and then compute from these measurements the magnitudes and directions of the principal strains ε_1 and ε_2. Such a group of strain gauges is called a strain rosette.

Strain Transformation Laws

If the components of strain at a point in a body are represented as $\varepsilon_x, \varepsilon_y$ and γ_{xy} with reference to the rectangular co-ordinate axes OX and OY, then the strain components with reference to a set of axes inclined at an angle θ with axis OX can be expressed as

$$\varepsilon_\theta = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \tag{3.25}$$

$$\gamma_\theta = (\varepsilon_y - \varepsilon_x)\sin 2\theta + \gamma_{xy}\cos 2\theta \tag{3.26}$$

and the principal strains are given by

$$\varepsilon_{max} \text{ or } \varepsilon_{min} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \tag{3.27}$$

The direction of the principal strains are defined by the angle θ as

$$\tan 2\theta = \left(\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}\right) \tag{3.28}$$

Also, the maximum shear strain at the point is given by following relation.

$$\gamma_{max} = \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} \tag{3.29}$$

Measurement of Strains using Rosettes

In a rectangular rosette, the strains are measured at angles denoted by $\theta_1 = 0, \theta_2 = 45^0$ and $\theta_3 = 90^0$. In an equiangular rosette (also called Delta Rosette)

$\theta_1 = 0^0$, $\theta_2 = 60^0$, $\theta_3 = 120^0$

Let $\varepsilon_{\theta_1}, \varepsilon_{\theta_2}$ and ε_{θ_3} be the strains measured at three different angles θ_1, θ_2 and θ_3 respectively. Now, using the section transformation laws, we can write the three simultaneous equations as follows:

$$\varepsilon_{\theta_1} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cos 2\theta_1 + \frac{\gamma_{xy}}{2}\sin 2\theta_1 \qquad \text{(a)}$$

$$\varepsilon_{\theta_2} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cos 2\theta_2 + \frac{\gamma_{xy}}{2}\sin 2\theta_2 \qquad \text{(b)}$$

$$\varepsilon_{\theta_3} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\cos 2\theta_3 + \frac{\gamma_{xy}}{2}\sin 2\theta_3 \qquad \text{(c)}$$

For a rectangular rosette,

$\theta_1 = 0, \quad \theta_2 = 45^0$ and $\theta_3 = 90^0$

Substituting the above in equations (a), (b) and (c),

We get

$$\varepsilon_0 = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right) + 0$$

$$= \frac{1}{2}\left(\varepsilon_x + \varepsilon_y + \varepsilon_x - \varepsilon_y\right)$$

$$\therefore \varepsilon_0 = \varepsilon_x$$

$$\varepsilon_{45} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)(0) + \frac{\gamma_{xy}}{2}$$

$$= \frac{1}{2}\left(\varepsilon_x + \varepsilon_y + \gamma_{xy}\right)$$

or $2\varepsilon_{45} = \varepsilon_x + \varepsilon_y + \gamma_{xy}$

$$\therefore \gamma_{xy} = 2\varepsilon_{45} - \left(\varepsilon_x + \varepsilon_y\right)$$

Also, $\varepsilon_{90} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) + \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\left(\cos 180^0\right) + \frac{\gamma_{xy}}{2}\left(\sin 180^0\right)$

$$= \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) - \left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)$$

$$= \frac{1}{2}\left(\varepsilon_x + \varepsilon_y - \varepsilon_x + \varepsilon_y\right)$$

$$\therefore \varepsilon_{90} = \varepsilon_y$$

Therefore, the components of strain are given by

$\varepsilon_x = \varepsilon_0, \quad \varepsilon_y = \varepsilon_{90^0}$ and $\gamma_{xy} = 2\varepsilon_{45} - \left(\varepsilon_0 + \varepsilon_{90}\right)$

For an equiangular rosette,

$\theta_1 = 0, \quad \theta_2 = 60^0, \quad \theta_3 = 120^0$

Substituting the above values in (a), (b) and (c), we get

$$\varepsilon_x = \varepsilon_0, \quad \varepsilon_y = \frac{1}{3}\left(2\varepsilon_{60} + 2\varepsilon_{120} - \varepsilon_0\right)$$

and $\gamma_{xy} = \frac{2}{\sqrt{3}}\left(\varepsilon_{60} - \varepsilon_{120}\right)$

Hence, using the values of $\varepsilon_{x,}\varepsilon_y$ and γ_{xy}, the principal strains $\varepsilon_{\max}$ and $\varepsilon_{\min}$ can be computed.

3.2.4 NUMERICAL EXAMPLES

Example 3.1

A sheet of metal is deformed uniformly in its own plane that the strain components related to a set of axes *xy* are

$\boldsymbol{\varepsilon_x = -200 \times 10^{-6}}$

$\boldsymbol{\varepsilon_y = 1000 \times 10^{-6}}$

$\boldsymbol{\gamma_{xy} = 900 \times 10^{-6}}$

(a) **Find the strain components associated with a set of axes $x'y'$ inclined at an angle of 30^o clockwise to the x y set as shown in the Figure 3.5. Also find the principal strains and the direction of the axes on which they act.**

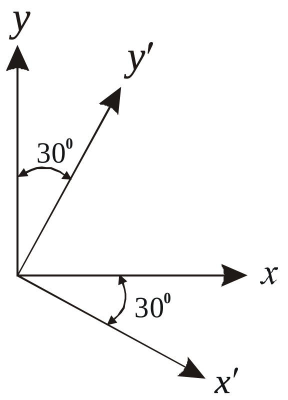

Figure 3.5

***Solution*: (*a*)**

The transformation equations for strains similar to that for stresses can be written as below:

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta$$

$$\frac{\gamma_{x'y'}}{2} = -\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta$$

Using Equation (3.19), we find

$$2\theta = \tan^{-1}\left(\frac{450}{600}\right) = 36.8^0$$

Radius of Mohr's circle = $R = \sqrt{(600)^2 + (450)^2} = 750$

Therefore,

$$\varepsilon_{x'} = \left(400\times10^{-6}\right) - \left(750\times10^{-6}\right)\cos\left(60^0 - 36.8^0\right)$$

$$= -290\times10^{-6}$$

$$\varepsilon_{y'} = \left(400\times10^{-6}\right) + \left(750\times10^{-6}\right)\cos\left(60^0 - 36.8^0\right)$$

$$=1090\times10^{-6}$$

Because point x' lies above the ε axis and point y' below ε axis, the shear strain $\gamma_{x'y'}$ is negative.

Therefore,

$$\frac{\gamma_{x'y'}}{2} = -\left(750\times10^{-6}\right)\sin\left(60^0 - 36.8^0\right)$$

$$= -295\times10^{-6}$$

hence, $\gamma_{x'y'} = -590\times10^{-6}$

Solution*: *(b)

From the Mohr's circle of strain, the Principal strains are

$\varepsilon_1 = 1150\times10^{-6}$

$\varepsilon_2 = -350\times10^{-6}$

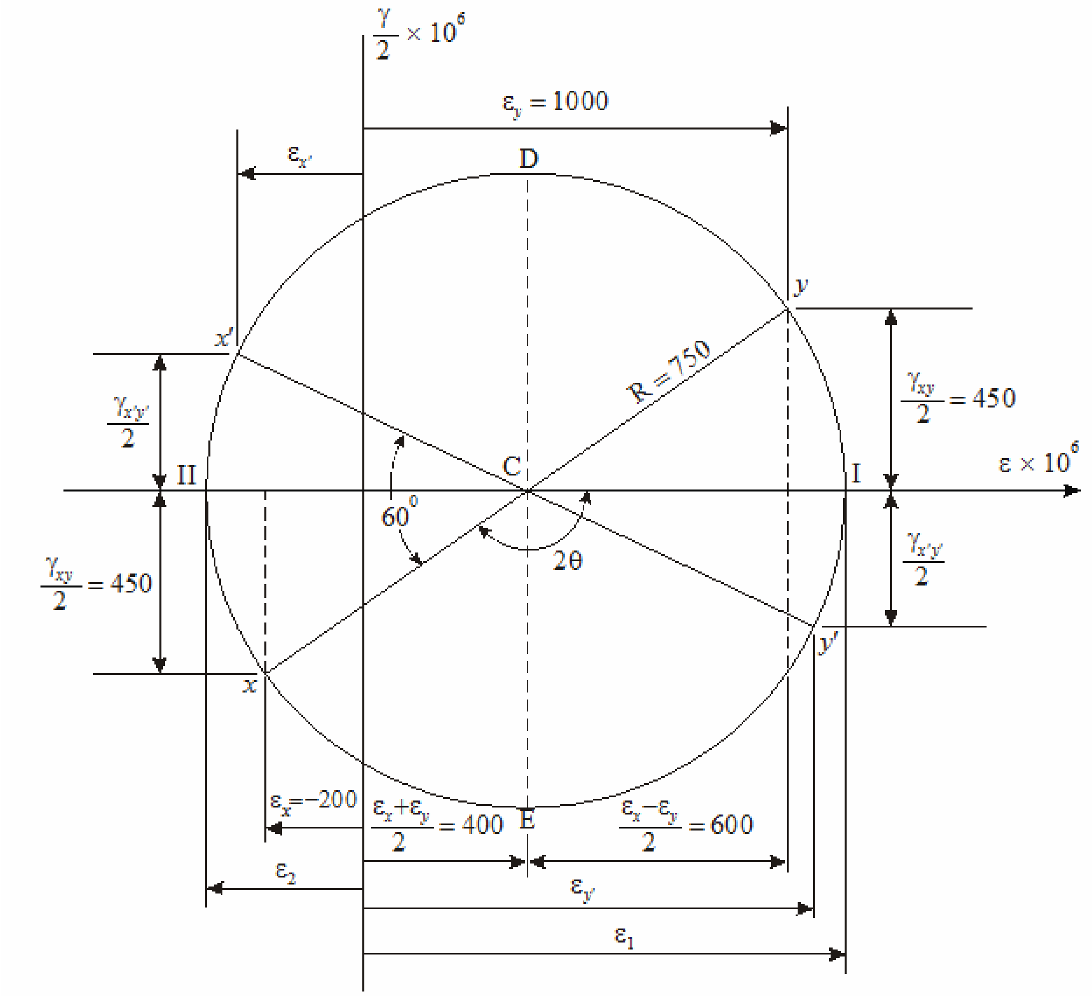

Figure 3.6 Construction of Mohr's strain circle

The directions of the principal axes of strain are shown in figure below.

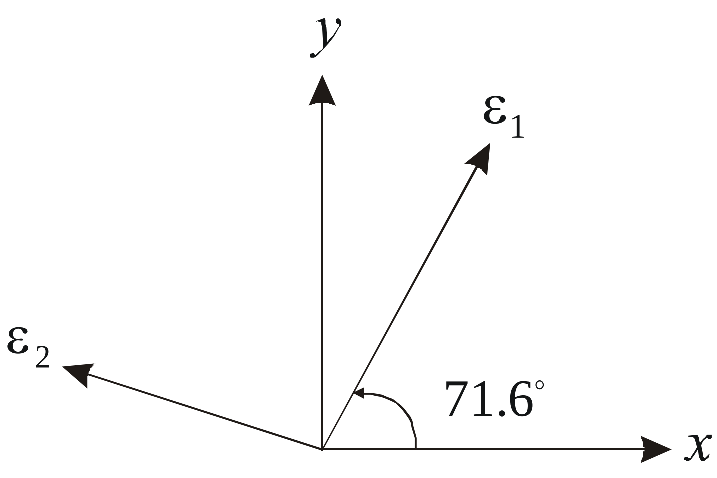

Figure 3.7

Example 3.2
By means of strain rosette, the following strains were recorded during the test on a structural member.

$$\varepsilon_0 = -13\times10^{-6}\,mm/mm,\ \varepsilon_{45} = 7.5\times10^{-6}\,mm/mm,\ \varepsilon_{90} = 13\times10^{-6}\,mm/mm$$

Determine (a) magnitude of principal strains
(b) Orientation of principal planes

***Solution*:** (a) We have for a rectangular strain rosette the following:

$\varepsilon_x = \varepsilon_0 \qquad \varepsilon_y = \varepsilon_{90} \qquad \gamma_{xy} = 2\varepsilon_{45} - (\varepsilon_0 + \varepsilon_{90})$

Substituting the values in the above relations, we get

$\varepsilon_x = -13\times10^{-6} \qquad \varepsilon_y = 13\times10^{-6}$

$\gamma_{xy} = 2\times7.5\times10^{-6} - (-12\times10^{-6} + 13\times10^{-6}) \qquad \therefore\ \gamma_{xy} = 15\times10^{-6}$

The principal strains can be determined from the following relation.

$$\varepsilon_{\max} \text{ or } \varepsilon_{\min} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}$$

$$\therefore \varepsilon_{\max} \text{ or } \varepsilon_{\min} = \left(\frac{-13+13}{2}\right)10^{-6} \pm \frac{1}{2}\sqrt{[(-13-13)10^{-6}]^2 + (15\times10^{-6})^2}$$

$$\therefore \varepsilon_{\max} \text{ or } \varepsilon_{\min} = \pm15\times10^{-6}$$

Hence $\varepsilon_{\max} = 15\times10^{-6}$ and $\varepsilon_{\min} = -15\times10^{-6}$

(b) The orientation of the principal strains can be obtained from the following relation

$$\tan 2\theta = \frac{\gamma_{xy}}{(\varepsilon_x - \varepsilon_y)}$$

$$= \frac{15\times10^{-6}}{(-13-13)10^{-6}}$$

$$\tan 2\theta = -0.577$$

$$\therefore 2\theta = 150^0$$

$$\therefore \theta = 75^0$$

Hence the directions of the principal planes are $\theta_1 = 75^0$ and $\theta_2 = 165^0$

Example 3.3

Data taken from a 45^0 strain rosette reads as follows:

$\varepsilon_0 = 750$ **micrometres/m**

$\varepsilon_{45} = -110$ **micrometres/m**

$\varepsilon_{90} = 210$ **micrometres/m**

Find the magnitudes and directions of principal strains.

Solution**:** Given $\varepsilon_0 = 750\times10^{-6}$

$\varepsilon_{45} = -110\times10^{-6}$

$\varepsilon_{90} = 210\times10^{-6}$

Now, for a rectangular rosette,

$$\varepsilon_x = \varepsilon_0 = 750\times10^{-6}$$

$$\varepsilon_y = \varepsilon_{90} = 210\times10^{-6}$$

$$\gamma_{xy} = 2\varepsilon_{45} - (\varepsilon_0 + \varepsilon_{90})$$

$$= 2(-110\times10^{-6}) - (750\times10^{-6} + 210\times10^{-6})$$

$$\gamma_{xy} = -1180\times10^{-6}$$

$\therefore$ The magnitudes of principal strains are

$$\varepsilon_{\max} \text{ or } \varepsilon_{\min} = \left(\frac{\varepsilon_x + \varepsilon_y}{2}\right) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2}$$

i.e., $\varepsilon_{\max}$ or $\varepsilon_{\min} = \left(\frac{750+210}{2}\right)10^{-6} \pm \frac{1}{2}\sqrt{\left[(750-210)10^{-6}\right]^2 + \left[(-1180)10^{-6}\right]^2}$

$$= 480\times10^{-6} \pm \frac{1}{2}(1297.7)10^{-6}$$

$$= 480\times10^{-6} \pm 648.85\times10^{-6}$$

$$\therefore \varepsilon_{\max} = \varepsilon_1 = 1128.85\times10^{-6}$$

$$\varepsilon_{\min} = \varepsilon_2 = -168.85\times10^{-6}$$

The directions of the principal strains are given by the relation

$$\tan 2\theta = \frac{\gamma_{xy}}{(\varepsilon_x - \varepsilon_y)}$$

$$\therefore \tan 2\theta = \frac{-1180\times10^{-6}}{(750-210)10^{-6}} = -2.185$$

$$\therefore 2\theta = 114.6^0$$

$$\therefore \theta_1 = 57.3^0 \text{ and } \theta_2 = 147.3^0$$

Example 3.4

If the displacement field in a body is specified as $u = (x^2+3)\,10^{-3}, v = 3y^2z\times10^{-3}$ and $w = (x+3z)\times10^{-3}$, determine the strain components at a point whose coordinates are (1,2,3)

***Solution*:** From Equation (3.3), we have

$$\varepsilon_x = \frac{\partial u}{\partial x} = 2x\times10^{-3},$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 6yz\times10^{-3},$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 3\times10^{-3}$$

$$\gamma_{xy} = \left[\frac{\partial}{\partial y}(x^2+3)\times10^{-3} + \frac{\partial}{\partial x}(3y^2z)\times10^{-3}\right]$$

$$\gamma_{xy} = 0$$

$$\gamma_{yz} = \left[\frac{\partial}{\partial z}\left(3y^2 z \times 10^{-3}\right) + \frac{\partial}{\partial y}\left(x + 3z\right) \times 10^{-3} \right]$$

$$\gamma_{yz} = 3y^2 \times 10^{-3}$$

and $\gamma_{zx} = \left[\frac{\partial}{\partial x}\left(x + 3z\right)10^{-3} + \frac{\partial}{\partial z}\left(x^2 + 3\right)10^{-3} \right]$

$$\gamma_{zx} = 1 \times 10^{-3}$$

Therefore at point (1, 2, 3), we get

$$\varepsilon_x = 2 \times 10^{-3}, \varepsilon_y = 6 \times 2 \times 3 \times 10^{-3} = 36 \times 10^{-3}, \varepsilon_z = 3 \times 10^{-3},$$

$$\gamma_{xy} = 0, \gamma_{yz} = 12 \times 10^{-3}, \gamma_{zx} = 1 \times 10^{-3}$$

Example 3.5

The strain components at a point with respect to *x y z* co-ordinate system are

$$\varepsilon_x = 0.10,\ \varepsilon_y = 0.20, \varepsilon_z = 0.30, \gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0.160$$

If the coordinate axes are rotated about the *z*-axis through 45^0 in the anticlockwise direction, determine the new strain components.

***Solution*:** Direction cosines

	x	y	z
x'	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
y'	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
z'	0	0	1

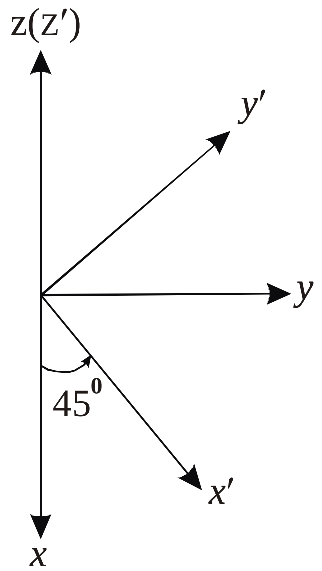

Figure 3.8

Here $l_1 = \frac{1}{\sqrt{2}}, \quad m_1 = -\frac{1}{\sqrt{2}}, \quad n_1 = 0$

$l_2 = \frac{1}{\sqrt{2}}, \quad m_2 = \frac{1}{\sqrt{2}}, \quad n_2 = 0$

$l_3 = 0, \quad m_3 = 0, \quad n_3 = 1$

Now, we have,

$$[\varepsilon'] = [a][\varepsilon][a]^T$$

$$[a][\varepsilon] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.1 & 0.08 & 0.08 \\ 0.08 & 0.2 & 0.08 \\ 0.08 & 0.08 & 0.3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.127 & 0.198 & 0.113 \\ -0.014 & 0.085 & 0 \\ 0.08 & 0.08 & 0.3 \end{bmatrix}$$

$$[\varepsilon'] = \begin{bmatrix} 0.127 & 0.198 & 0.113 \\ -0.014 & 0.085 & 0 \\ 0.08 & 0.08 & 0.3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\varepsilon'] = \begin{bmatrix} 0.23 & 0.05 & 0.113 \\ 0.05 & 0.07 & 0 \\ 0.113 & 0.3 & 0.3 \end{bmatrix}$$

Therefore, the new strain components are

$$\varepsilon_x = 0.23, \quad \varepsilon_y = 0.07, \quad \varepsilon_z = 0.3$$

$$\frac{1}{2}\gamma_{xy} = 0.05 \text{ or } \gamma_{xy} = 0.05 \times 2 = 0.1$$

$$\gamma_{yz} = 0, \quad \gamma_{zx} = 0.113 \times 2 = 0.226$$

Example 3.6

The components of strain at a point in a body are as follows:

$$\varepsilon_x = 0.1, \quad \varepsilon_y = -0.05, \quad \varepsilon_z = 0.05, \quad \gamma_{xy} = 0.3, \quad \gamma_{yz} = 0.1, \quad \gamma_{xz} = -0.08$$

Determine the principal strains and the principal directions.

Solution: The strain tensor is given by

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{xy}}{2} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} 0.1 & 0.15 & -0.04 \\ 0.15 & -0.05 & 0.05 \\ -0.04 & 0.05 & 0.05 \end{bmatrix}$$

The invariants of strain tensor are

$$J_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z = 0.1 - 0.05 + 0.05 = 0.1$$

$$J_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \frac{1}{4}\left(\gamma^2{}_{xy} + \gamma^2{}_{yz} + \gamma^2{}_{zx}\right)$$

$$= (0.1)(-0.05) + (-0.05)(0.05) + (0.05)(0.1) - \frac{1}{4}\left[(0.3)^2 + (0.1)^2 + (-0.08)^2\right]$$

$$\therefore J_2 = -0.0291$$

$$\mathrm{J}_3 = (0.1)(-0.05)(0.05) + \frac{1}{4}\left[(0.3)(0.1)(-0.08) - 0.1(0.1)^2 + 0.05(0.08)^2 - 0.05(0.3)^2\right]$$

$$J_3 = -0.002145$$

$\therefore$ The cubic equation is

$$\varepsilon^3 - 0.1\varepsilon^2 - 0.0291\varepsilon + 0.002145 = 0 \qquad \text{(i)}$$

Now $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Or $\cos^3\theta - \frac{3}{4}\cos\theta - \frac{1}{4}\cos 3\theta = 0$ (ii)

Let $\varepsilon = r\cos\theta + \frac{J_1}{3}$

$$= r\cos\theta + \frac{0.1}{3}$$

$$\varepsilon = r\cos\theta + 0.033$$

$\therefore$ (i) can be written as

$$(r\cos\theta + 0.033)^3 - 0.1(r\cos\theta + 0.033)^2 - 0.0291(r\cos\theta + 0.033) + 0.002145 = 0$$

$$(r\cos\theta + 0.033)(r\cos\theta + 0.033)^2 - 0.1(r\cos\theta + 0.033)^2 - 0.0291 r\cos\theta$$
$$- 0.00096 + 0.002145 = 0$$

$$(r\cos\theta + 0.033)\left(r^2\cos^2\theta + 0.067r\cos\theta + 0.00109\right)$$
$$-0.1\left(r^2\cos^2\theta + 0.067r\cos\theta + 0.00109\right) - 0.0291r\cos\theta - 0.00096 + 0.002145 = 0$$

$$r^3\cos^3\theta + 0.067r^2\cos^2\theta + 0.00109r\cos\theta + 0.033r^2\cos^2\theta + 0.0022r\cos\theta +$$
$$0.000036 - 0.1r^2\cos^2\theta - 0.0067r\cos\theta - 0.000109 - 0.0291r\cos\theta - 0.00096 +$$
$$0.002145 = 0$$

i.e., $r^3\cos^3\theta - 0.03251r\cos\theta - 0.00112 = 0$

or $\cos^3\theta - \dfrac{0.03251}{r^2}\cos\theta - \dfrac{0.00112}{r^3} = 0$ (iii)

Hence Equations (ii) and (iii) are identical if

$$\frac{0.03251}{r^2} = \frac{3}{4}$$

i.e., $r = \sqrt{\dfrac{4\times 0.03251}{3}} = 0.2082$

and $\dfrac{\cos 3\theta}{4} = \dfrac{0.00112}{r^3}$

or $\cos 3\theta = \dfrac{4\times 0.00112}{(0.2082)^3} = 0.496 \cong 0.5$

$\therefore 3\theta = 60^0$ or $\theta_1 = \dfrac{60}{3} = 20^0$

$\theta_2 = 100^0$ $\theta_3 = 140^0$

$$\therefore \varepsilon_1 = r_1\cos\theta_1 + \frac{J_1}{3}$$
$$= 0.2082\cos 20^0 + \frac{0.1}{3}$$
$$\varepsilon_1 = 0.228$$

$$\varepsilon_2 = r_2\cos\theta_2 + \frac{J_1}{3} = 0.2082\cos 100^0 + \frac{0.1}{3} = -0.0031$$
$$\varepsilon_3 = r_3\cos\theta_3 + \frac{J_1}{3} = 0.2082\cos 140^0 + \frac{0.1}{3} = -0.126$$

To find principal directions

(a) Principal direction for ε_1

$$\begin{bmatrix} (0.1-\varepsilon_1) & 0.15 & -0.04 \\ 0.15 & (-0.05-\varepsilon_1) & 0.05 \\ -0.04 & 0.05 & (0.05-\varepsilon_1) \end{bmatrix}$$

$$= \begin{bmatrix} (0.1-0.228) & 0.15 & -0.04 \\ 0.15 & (-0.05-0.228) & 0.05 \\ -0.04 & 0.05 & (0.05-0.228) \end{bmatrix}$$

$$= \begin{bmatrix} -0.128 & 0.15 & -0.04 \\ 0.15 & -0.278 & 0.05 \\ -0.04 & 0.05 & -0.178 \end{bmatrix}$$

Now, $A_1 = \begin{vmatrix} -0.278 & 0.05 \\ 0.05 & -0.178 \end{vmatrix} = (-0.278)(-0.178) - (0.05)(0.05)$

$\therefore A_1 = 0.046984$

$B_1 = -\begin{vmatrix} 0.15 & 0.05 \\ -0.04 & -0.178 \end{vmatrix} = -[0.15 \times (-0.178) + (0.05)(0.04)]$

$\therefore B_1 = 0.0247$

$C_1 = \begin{vmatrix} 0.15 & -0.278 \\ -0.04 & 0.05 \end{vmatrix} = 0.15 \times 0.05 - 0.278 \times 0.04$

$\therefore C_1 = -0.00362$

$$\sqrt{A_1^2 + B_1^2 + C_1^2} = \sqrt{(0.046984)^2 + (0.0247)^2 + (-0.00362)^2}$$

$$= 0.0532$$

$$\therefore l_1 = \frac{A_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{0.046984}{0.0532} = 0.883$$

$$m_1 = \frac{B_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{0.0247}{0.0532} = 0.464$$

$$n_1 = \frac{C_1}{\sqrt{A_1^{\,2} + B_1^{\,2} + C_1^{\,2}}} = \frac{-0.00362}{0.0532} = -0.068$$

Similarly, the principal directions for ε_2 can be determined as follows:

$$\begin{bmatrix} (0.1 + 0.0031) & 0.15 & -0.04 \\ 0.15 & (-0.05 + 0.0031) & 0.05 \\ -0.04 & 0.05 & (0.05 + 0.0031) \end{bmatrix}$$

$$= \begin{bmatrix} 0.1031 & 0.15 & -0.04 \\ 0.15 & -0.0469 & 0.05 \\ -0.04 & 0.05 & 0.0531 \end{bmatrix}$$

$$A_2 = \begin{vmatrix} -0.0469 & 0.05 \\ 0.05 & 0.0531 \end{vmatrix} = -0.00249 - 0.0025 = -0.00499$$

$$B_2 = \begin{vmatrix} 0.15 & 0.05 \\ -0.04 & 0.0531 \end{vmatrix} = -(0.007965 + 0.002) = -0.009965$$

$$C_2 = \begin{vmatrix} 0.15 & -0.0469 \\ -0.04 & 0.05 \end{vmatrix} = 0.0075 - 0.00188 = 0.00562$$

Now, $\sqrt{A_2^{\,2} + B_2^{\,2} + C_2^{\,2}} = \sqrt{(-0.00499)^2 + (-0.009965)^2 + (0.00562)^2} = 0.0125$

$$\therefore l_2 = \frac{A_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}} = \frac{-0.00499}{0.0125} = -0.399$$

$$m_2 = \frac{B_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}} = \frac{-0.009965}{0.0125} = -0.797$$

$$n_2 = \frac{C_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}} = \frac{0.00562}{0.0125} = 0.450$$

And for $\varepsilon_3 = -0.126$

$$\begin{vmatrix} (0.1 + 0.126) & 0.15 & -0.04 \\ 0.15 & (-0.05 + 0.126) & 0.05 \\ -0.04 & 0.05 & (0.05 + 0.126) \end{vmatrix}$$

$$= \begin{vmatrix} 0.226 & 0.15 & -0.04 \\ 0.15 & 0.076 & 0.05 \\ -0.04 & 0.05 & 0.176 \end{vmatrix}$$

Now, $A_3 = \begin{vmatrix} 0.076 & 0.05 \\ 0.05 & 0.176 \end{vmatrix} = 0.0134 - 0.0025 = 0.0109$

$$B_3 = -\begin{vmatrix} 0.15 & 0.05 \\ -0.04 & 0.176 \end{vmatrix} = -(0.0264 + 0.002) = -0.0284$$

$$C_3 = -\begin{vmatrix} 0.15 & 0.076 \\ -0.04 & 0.05 \end{vmatrix} = 0.0075 + 0.00304 = 0.01054$$

Now, $\sqrt{A_3^{\ 2} + B_3^{\ 2} + C_3^{\ 2}} = \sqrt{(0.0109)^2 + (-0.0284)^2 + (0.01054)^2} = 0.0322$

$$\therefore l_3 = \frac{A_3}{\sqrt{A_3^2 + B_3^2 + C_3^2}} = \frac{0.0109}{0.0322} = 0.338$$

$$m_3 = \frac{B_3}{\sqrt{A_3^2 + B_3^2 + C_3^2}} = \frac{-0.0284}{0.0322} = -0.882$$

$$n_3 = \frac{C_3}{\sqrt{A_3^2 + B_3^2 + C_3^2}} = \frac{0.01054}{0.0322} = 0.327$$

Example 3.7

The displacement components in a strained body are as follows:

$u = 0.01xy + 0.02y^2, v = 0.02x^2 + 0.01z^3 y, w = 0.01xy^2 + 0.05z^2$

Determine the strain matrix at the point *P* (3,2, -5)

***Solution*:** $\varepsilon_x = \frac{\partial u}{\partial x} = 0.01y$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 0.01z^3$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0.1z$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0.04x + 0.01x + 0.04y$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0.02xy + 0.03z^2 y$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 + 0.01y^2$$

At point *P* (3, 2, -5), the strain components are

$\varepsilon_x = 0.02, \quad \varepsilon_y = -1.25, \quad \varepsilon_z = -0.5$

$\gamma_{xy} = 0.23, \quad \gamma_{yz} = 1.62, \quad \gamma_{zx} = 0.04$

Now, the strain tensor is given by

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix}$$

$\therefore$ Strain matrix becomes

$$\varepsilon_{ij} = \begin{bmatrix} 0.02 & 0.115 & 0.02 \\ 0.115 & -1.25 & 0.81 \\ 0.02 & 0.81 & -0.50 \end{bmatrix}$$

Example 3.8

The strain tensor at a point in a body is given by

$$\varepsilon_{ij} = \begin{bmatrix} 0.0001 & 0.0002 & 0.0005 \\ 0.0002 & 0.0003 & 0.0004 \\ 0.0005 & 0.0004 & 0.0005 \end{bmatrix}$$

Determine (a) octahedral normal and shearing strains. (b) Deviator and Spherical strain tensors.

Solution**:** For the octahedral plane, the direction cosines are $l = m = n = \frac{1}{\sqrt{3}}$

(a) octahedral normal strain is given by

$$(\varepsilon_n)_{oct} = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + 2(\varepsilon_{xy} lm + \varepsilon_{yz} mn + \varepsilon_{zx} nl)$$

Here $\varepsilon_{xy} = \frac{1}{2}\gamma_{xy}$, $\varepsilon_{yz} = \frac{1}{2}\gamma_{yz}$ and $\varepsilon_{zx} = \frac{1}{2}\gamma_{zx}$

$$\therefore (\varepsilon_n)_{oct} = 0.0001\left(\frac{1}{\sqrt{3}}\right)^2 + 0.0003\left(\frac{1}{\sqrt{3}}\right)^2 + 0.0005\left(\frac{1}{\sqrt{3}}\right)^2 +$$

$$2\left[0.0002\left(\frac{1}{3}\right) + 0.0004\left(\frac{1}{3}\right) + 0.0005\left(\frac{1}{3}\right)\right]$$

$$\therefore (\varepsilon_n)_{oct} = 0.001$$

Octahedral Shearing Strain is given by

$$\gamma_{oct} = 2\sqrt{(\varepsilon_R)^2_{oct} - (\varepsilon_n)^2_{oct}}$$

where $(\varepsilon_R)_{oct}$ = Resultant strain on octahedral plane

$$\therefore (\varepsilon_R)_{oct} = \sqrt{\frac{1}{3}\left[(\varepsilon_x + \varepsilon_{xy} + \varepsilon_{xz})^2 + (\varepsilon_{xy} + \varepsilon_y + \varepsilon_{yz})^2 + (\varepsilon_{xz} + \varepsilon_{yz} + \varepsilon_y)^2\right]}$$

$$= \sqrt{\frac{1}{3}\left[(0.0001 + 0.0002 + 0.0005)^2 + (0.0002 + 0.0003 + 0.0004)^2 (0.0005 + 0.0004 + 0.0005)^2\right]}$$

$$\therefore (\varepsilon_R)_{oct} = 0.001066$$

$$\therefore \gamma_{oct} = 2\sqrt{(0.00106)^2 - (0.001)^2}$$

$$\therefore \gamma_{oct} = 0.000739$$

(b) Deviator and Spherical strain tensors.

Here Mean Strain = $\varepsilon_m = \dfrac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3}$

$$= \frac{0.0001 + 0.0003 + 0.0005}{3}$$

$$\therefore \varepsilon_m = 0.0003$$

$$\therefore \text{Deviator Strain tensor} = \begin{bmatrix} (0.0001 - 0.0003) & 0.0002 & 0.0005 \\ 0.0002 & (0.0003 - 0.0003) & 0.0004 \\ 0.0005 & 0.0004 & (0.0005 - 0.0003) \end{bmatrix}$$

i.e.,
$$E' = \begin{bmatrix} -0.0002 & 0.0002 & 0.0005 \\ 0.0002 & 0 & 0.0004 \\ 0.0005 & 0.0004 & 0.0002 \end{bmatrix}$$

and spherical strain tensor = $E'' = \begin{bmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{bmatrix}$

i.e.,
$$E'' = \begin{bmatrix} 0.0003 & 0 & 0 \\ 0 & 0.0003 & 0 \\ 0 & 0 & 0.0003 \end{bmatrix}$$

Example 3.9

The components of strain at a point in a body are as follows:

$$\varepsilon_x = c_1 z(x^2 + y^2)$$

$$\varepsilon_y = x^2 z$$

$$\gamma_{xy} = 2c_2 xyz$$

where c_1 and c_2 are constants. Check whether the strain field is compatible one?

***Solution*:** For the compatibility condition of the strain field, the system of strains must satisfy the compatibility equations

i.e., $\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$

Now, using the given strain field,

$$\frac{\partial \varepsilon_x}{\partial y} = 2c_1 yz, \quad \frac{\partial^2 \varepsilon_x}{\partial y^2} = 2c_1 z$$

$$\frac{\partial \varepsilon_y}{\partial x} = 2xz, \quad \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2z$$

$$\frac{\partial \gamma_{xy}}{\partial x} = 2c_2 yz, \quad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 2c_2 z$$

$\therefore \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 2c_1 z + 2z = 2z(1 + c_1)$ and $\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 2c_2 z$

Since $\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} \neq \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$, the strain field is not compatible.

Example 3.10

Under what conditions are the following expressions for the components of strain at a point compatible?

$$\varepsilon_x = 2axy^2 + by^2 + 2cxy$$

$$\varepsilon_y = ax^2 + bx$$

$$\gamma_{xy} = \alpha x^2 y + \beta xy + ax^2 + \eta y$$

***Solution*:** For compatibility, the strain components must satisfy the compatibility equation.

i.e., $\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$ (i)

or $\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$ (ii)

Now, $\varepsilon_x = 2axy^2 + by^2 + 2cxy$

$$\therefore \frac{\partial \varepsilon_x}{\partial y} = 4axy + 2by + 2cx$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = 4ax + 2b$$

$$\varepsilon_y = ax^2 + bx$$

$$\frac{\partial \varepsilon_y}{\partial x} = 2ax + b$$

$$\frac{\partial^2 \varepsilon_y}{\partial x^2} = 2a$$

$$\gamma_{xy} = \alpha x^2 y + \beta xy + ax^2 + \eta y$$

$$\frac{\partial \gamma_{xy}}{\partial x} = 2\alpha xy + \beta y + 2ax$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 2\alpha x + \beta$$

$\therefore$ (i) becomes

$$4ax + 2b + 2a = 2\alpha x + \beta$$

$$4ax + 2(a + b) = 2\alpha x + \beta$$

$\therefore 4ax = 2\alpha x$
or $\alpha = 2a$
and $\beta = 2(a + b)$

Example 3.11
For the given displacement field
$u = c(x^2 + 2x)$
$v = c(4x + 2y^2 + z)$
$w = 4cz^2$
where c is a very small constant, determine the strain at (2,1,3), in the direction 0, $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

***Solution*:** $\varepsilon_x = \frac{\partial u}{\partial x} = 2cx, \quad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 4c + 0 = 4c$

$$\varepsilon_y = \frac{\partial v}{\partial y} = 4cy, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 + c = c$$

$$\varepsilon_z = \frac{\partial w}{\partial z} = 8cz, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 2c + 0 = 2c$$

$\therefore$ At point (2,1,3),

$\varepsilon_x = 2c \times 2 = 4c, \quad \gamma_{xy} = 4c$

$\varepsilon_y = 4c \times 1 = 4c, \quad \gamma_{yz} = c$

$\varepsilon_z = 8c \times 3 = 24c, \quad \gamma_{zx} = 2c$

$\therefore$ The Resultant strain in the direction $l = 0,\ m = -\frac{1}{\sqrt{2}},\ n = \frac{1}{\sqrt{2}}$ is given by

$$\varepsilon_r = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{zx} nl$$

$$= 0 + 4c\left(-\frac{1}{\sqrt{2}}\right)^2 + 24c\left(\frac{1}{\sqrt{2}}\right)^2 + 4c(0) + c\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 2c(0)$$

$$\therefore \varepsilon_r = 13.5c$$

Example 3.12

The strain components at a point are given by

$\varepsilon_x = 0.01,\ \varepsilon_y = -0.02,\ \varepsilon_z = 0.03,\ \gamma_{xy} = 0.015,\ \gamma_{yz} = 0.02,\ \gamma_{xz} = -0.01$

Determine the normal and shearing strains on the octahedral plane.

***Solution*:** An octahedral plane is one which is inclined equally to the three principal co-ordinates. Its direction cosines are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Now, the normal strain on the octahedral plane is

$$(\varepsilon_n)_{oct} = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{zx} nl$$

$$= \frac{1}{3}[0.01 - 0.02 + 0.03 + 0.015 + 0.02 - 0.01]$$

$$\therefore (\varepsilon_n)_{oct} = 0.015$$

The strain tensor can be written as

$$\begin{pmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{pmatrix} = \begin{pmatrix} 0.01 & \frac{0.015}{2} & -\frac{0.01}{2} \\ \frac{0.015}{2} & -0.02 & \frac{0.02}{2} \\ -\frac{0.01}{2} & \frac{0.02}{2} & 0.03 \end{pmatrix} = \begin{pmatrix} 0.01 & 0.0075 & -0.005 \\ 0.0075 & -0.02 & 0.01 \\ -0.005 & 0.01 & 0.03 \end{pmatrix}$$

Now, the resultant strain on the octahedral plane is given by

$$(\varepsilon_R)_{oct} = \sqrt{\frac{1}{3}\left\{(\varepsilon_x + \varepsilon_{xy} + \varepsilon_{xz})^2 + (\varepsilon_{xy} + \varepsilon_y + \varepsilon_{yz})^2 + (\varepsilon_{xz} + \varepsilon_{yz} + \varepsilon_z)^2\right\}}$$

$$= \sqrt{\frac{1}{3}\left\{(0.01 + 0.0075 - 0.005)^2 + (0.0075 - 0.02 + 0.01)^2 + (-0.005 + 0.01 + 0.03)^2\right\}}$$

$$= \sqrt{0.0004625}$$

$\therefore (\varepsilon_R)_{oct} = 0.0215$

and octahedral shearing strain is given by

$$(\varepsilon_S)_{oct} = 2\sqrt{(\varepsilon_R)^2 - (\varepsilon_n)^2} = 2\sqrt{(0.0215)^2 - (0.015)^2}$$

$\therefore (\varepsilon_S)_{oct} = 0.031$

Example 3.13

The displacement field is given by

$$u = K(x^2 + 2z), \quad v = K(4x + 2y^2 + z), \quad w = 4Kz^2$$

where *K* is a very small constant. What are the strains at (2,2,3) in directions

$$(a)\, l = 0, \; m = \frac{1}{\sqrt{2}}, \; n = \frac{1}{\sqrt{2}}, \quad (b)\, l = 1, \; m = n = 0, \quad (c)\, l = 0.6, \; m = 0, \; n = 0.8$$

***Solution*:** $\varepsilon_x = \frac{\partial u}{\partial x} = 2Kx, \quad \varepsilon_y = \frac{\partial v}{\partial y} = 4Ky, \quad \varepsilon_z = \frac{\partial w}{\partial z} = 8Kz$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 4K + 0 = 4K$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 + K = K$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 2K + 0 = 2K$$

$\therefore$ At point (2,2,3),

$$\varepsilon_x = 4K, \quad \varepsilon_y = 8K, \quad \varepsilon_z = 24K$$

$$\gamma_{xy} = 4K, \quad \gamma_{yz} = K, \quad \gamma_{zx} = 2K$$

Now, the strain in any direction is given by

$$\varepsilon_r = \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + \gamma_{xy} lm + \gamma_{yz} mn + \gamma_{zx} nl \qquad \text{(i)}$$

Case (a)

Substituting the values in expression (i), we get

$$\varepsilon_r = 4K(0) + 8K\left(\frac{1}{\sqrt{2}}\right)^2 + 24K\left(\frac{1}{\sqrt{2}}\right)^2 + 4K(0) + K\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + 2K(0)$$

$$\therefore \varepsilon_r = 4K + 12K + \frac{1}{2}K$$

$$\therefore \varepsilon_r = 16.5K$$

Case (b)

$$\varepsilon_r = 4K(1)^2 + 8K(0) + 24(0) + 4K(0) + K(0) + 2K(0)$$

$$\therefore \varepsilon_r = 4K$$

Case (c)

$$\varepsilon_r = 4K(0.6)^2 + 8K(0) + 24(0.8)^2 + 4K(0) + K(0) + 2K(0.8)(0.6)$$

$$\therefore \varepsilon_r = 17.76K$$