# Module 2: Analysis of Stress

## 2.1.1 INTRODUCTION

A body under the action of external forces, undergoes distortion and the effect due to this system of forces is transmitted throughout the body developing internal forces in it. To examine these internal forces at a point O in Figure 2.1 (a), inside the body, consider a plane MN passing through the point O. If the plane is divided into a number of small areas, as in the Figure 2.1 (b), and the forces acting on each of these are measured, it will be observed that these forces vary from one small area to the next. On the small area $\Delta A$ at point O, a force $\Delta F$ will be acting as shown in the Figure 2.1 (b). From this the concept of stress as the internal force per unit area can be understood. Assuming that the material is continuous, the term "stress" at any point across a small area $\Delta A$ can be defined by the limiting equation as below.
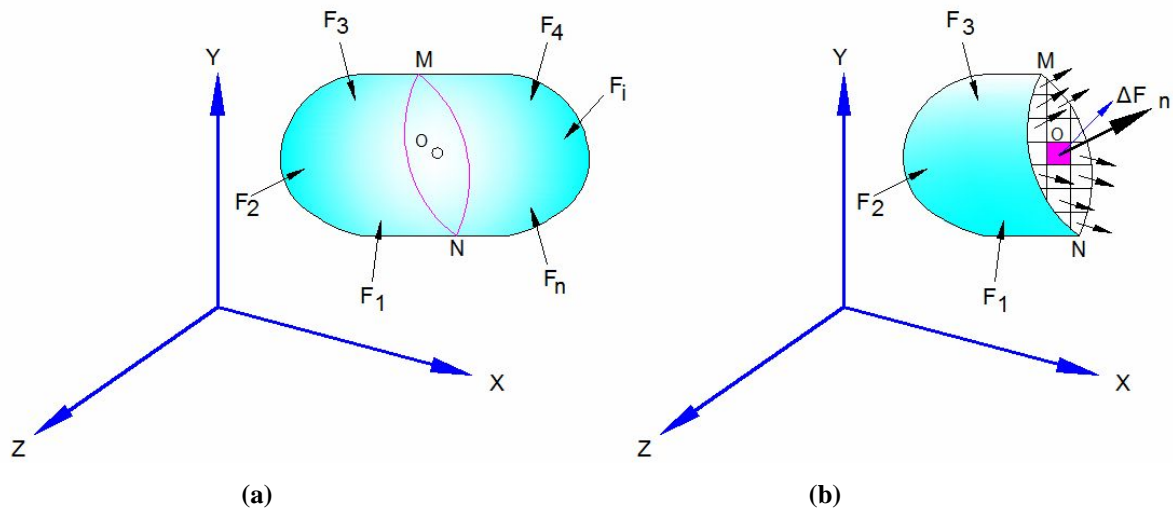


**Figure 2.1 Forces acting on a body**

$$\text{Stress} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{2.0}$$

where $\Delta F$ is the internal force on the area $\Delta A$ surrounding the given point. Stress is sometimes referred to as force intensity.

## 2.1.2 NOTATION OF STRESS

Here, a single suffix for notation $\sigma$, like $\sigma_x, \sigma_y, \sigma_z$, is used for the direct stresses and double suffix for notation is used for shear stresses like $\tau_{xy}, \tau_{xz}$, etc. $\tau_{xy}$ means a stress, produced by an internal force in the direction of Y, acting on a surface, having a normal in the direction of X.

## 2.1.3 CONCEPT OF DIRECT STRESS AND SHEAR STRESS
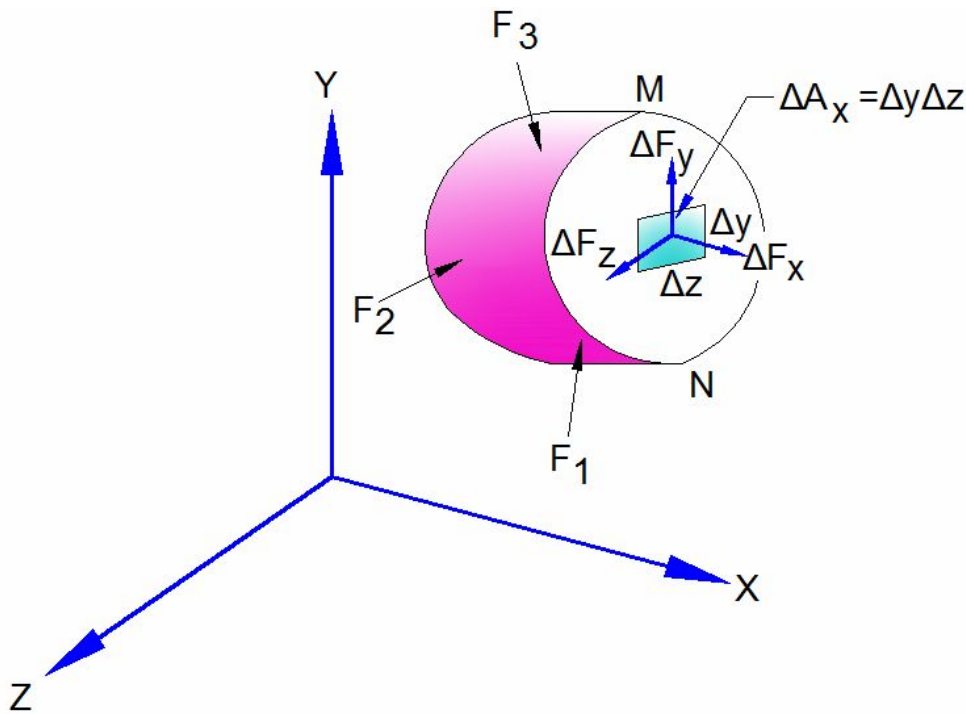


**Figure 2.2 Force components of $\Delta F$ acting on small area centered on point O**

Figure 2.2 shows the rectangular components of the force vector $\Delta F$ referred to corresponding axes. Taking the ratios $\dfrac{\Delta F_x}{\Delta A_x}, \dfrac{\Delta F_y}{\Delta A_x}, \dfrac{\Delta F_z}{\Delta A_x}$, we have three quantities that establish the average intensity of the force on the area $\Delta A_x$. In the limit as $\Delta A \rightarrow 0$, the above ratios define the force intensity acting on the X-face at point O. These values of the three intensities are defined as the "Stress components" associated with the X-face at point O. The stress components parallel to the surface are called "Shear stress components" denoted by $\tau$. The shear stress component acting on the X-face in the *y*-direction is identified as $\tau_{xy.}$ The stress component perpendicular to the face is called "Normal Stress" or "Direct stress" component and is denoted by $\sigma$. This is identified as $\sigma_x$ along X-direction.

From the above discussions, the stress components on the X -face at point O are defined as follows in terms of force intensity ratios

$$\sigma_x = \lim_{\Delta A_x \to 0} \frac{\Delta F_x}{\Delta A_x}$$

$$\tau_{xy} = \lim_{\Delta A_x \to 0} \frac{\Delta F_y}{\Delta A_x}$$

(2.1)

$$\tau_{xz} = \lim_{\Delta A_x \to 0} \frac{\Delta F_z}{\Delta A_x}$$

The above stress components are illustrated in the Figure 2.3 below.
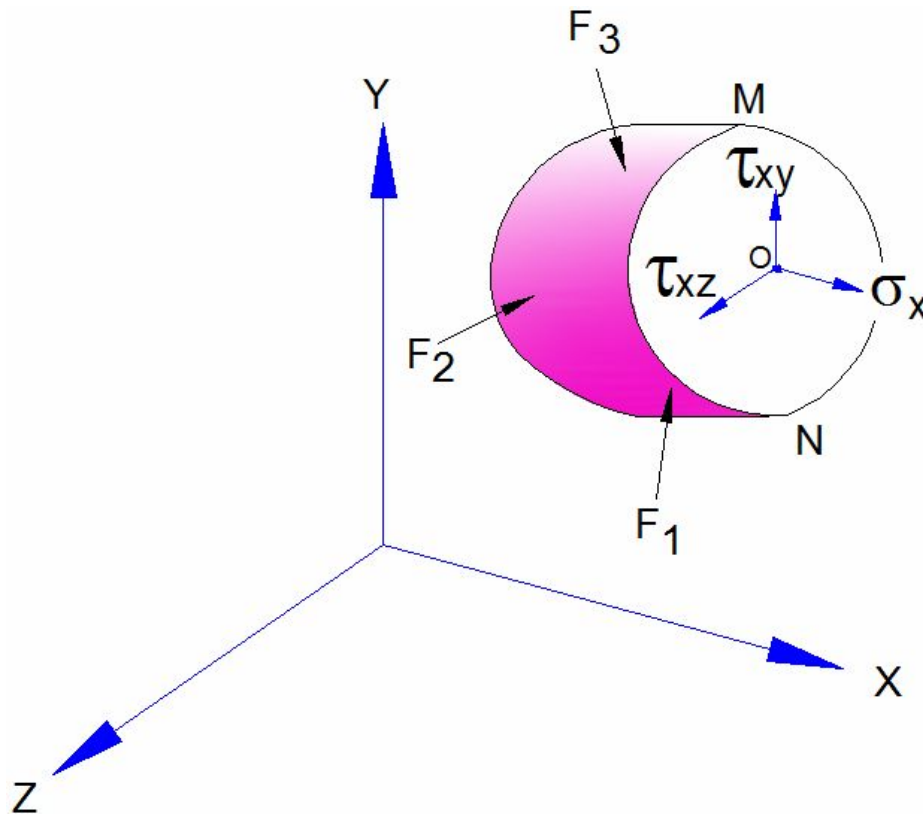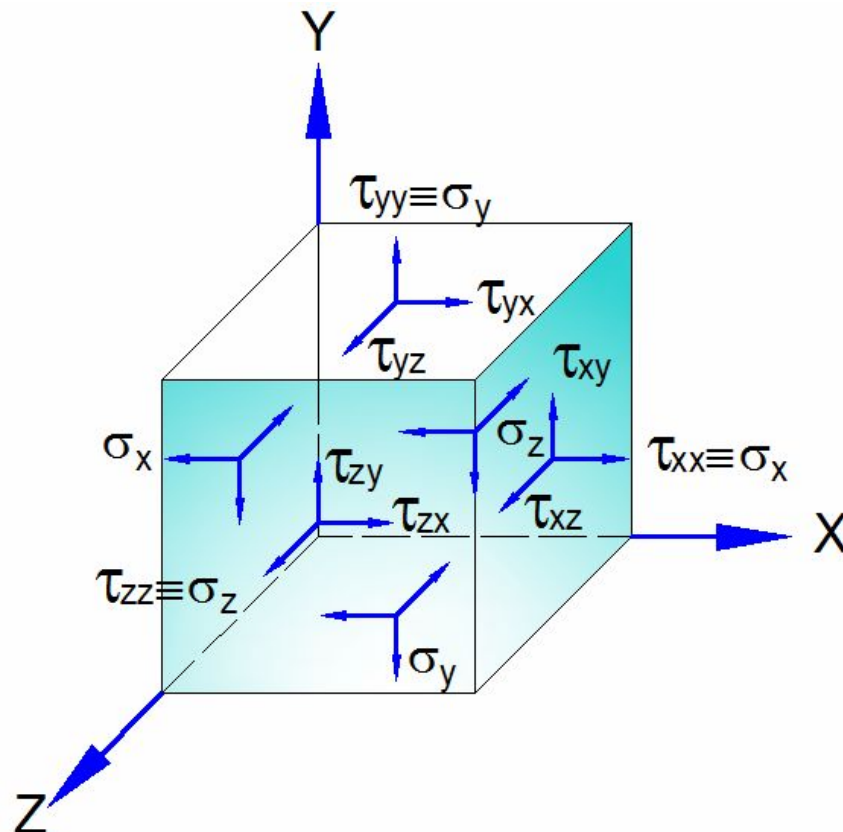


**Figure 2.3 Stress components at point O**

## 2.1.4   STRESS TENSOR

Let O be the point in a body shown in Figure 2.1 (a).  Passing through that point, infinitely many planes may be drawn.  As the resultant forces acting on these planes is the same, the stresses on these planes are different because the areas and the inclinations of these planes are different.  Therefore, for a complete description of stress, we have to specify not only its magnitude, direction and sense but also the surface on which it acts. For this reason, the stress is called a "Tensor".



**Figure 2.4 Stress components acting on parallelopiped**

Figure 2.4 depicts three-orthogonal co-ordinate planes representing a parallelopiped on which are nine components of stress. Of these three are direct stresses and six are shear stresses. In tensor notation, these can be expressed by the tensor $\tau_{ij}$, where $i = x, y, z$ and $j = x, y, z$.

In matrix notation, it is often written as

$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \tag{2.2}$$

It is also written as

$$S = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{2.3}$$

## 2.1.5 SPHERICAL AND DEVIATORIAL STRESS TENSORS

A general stress-tensor can be conveniently divided into two parts as shown above. Let us now define a new stress term ($\sigma_m$) as the mean stress, so that

$$\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \tag{2.4}$$

Imagine a hydrostatic type of stress having all the normal stresses equal to $\sigma_m$, and all the shear stresses are zero. We can divide the stress tensor into two parts, one having only the "hydrostatic stress" and the other, "deviatorial stress". The hydrostatic type of stress is given by

$$\begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} \tag{2.5}$$

The deviatorial type of stress is given by

$$\begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_m \end{bmatrix} \tag{2.6}$$

Here the hydrostatic type of stress is known as "spherical stress tensor" and the other is known as the "deviatorial stress tensor".

It will be seen later that the deviatorial part produces changes in shape of the body and finally causes failure. The spherical part is rather harmless, produces only uniform volume changes without any change of shape, and does not necessarily cause failure.

## 2.1.6  INDICIAL NOTATION

An alternate notation called index or indicial notation for stress is more convenient for general discussions in elasticity. In indicial notation, the co-ordinate axes $x$, $y$ and $z$ are replaced by numbered axes $x_1$, $x_2$ and $x_3$ respectively. The components of the force $\Delta F$ of Figure 2.1 (a) is written as $\Delta F_1$, $\Delta F_2$ and $\Delta F_3$, where the numerical subscript indicates the component with respect to the numbered coordinate axes.

The definitions of the components of stress acting on the face $x_1$ can be written in indicial form as follows:

$$\sigma_{11} = \lim_{\Delta A_1 \to 0} \frac{\Delta F_1}{\Delta A_1}$$

$$\sigma_{12} = \lim_{\Delta A_1 \to 0} \frac{\Delta F_2}{\Delta A_1} \qquad (2.7)$$

$$\sigma_{13} = \lim_{\Delta A_1 \to 0} \frac{\Delta F_3}{\Delta A_1}$$

Here, the symbol $\sigma$ is used for both normal and shear stresses. In general, all components of stress can now be defined by a single equation as below.

$$\sigma_{ij} = \lim_{\Delta A_i \to 0} \frac{\Delta F_j}{\Delta A_i} \qquad (2.8)$$

Here $i$ and $j$ take on the values 1, 2 or 3.

## 2.1.7  TYPES OF STRESS

Stresses may be classified in two ways, i.e., according to the type of body on which they act, or the nature of the stress itself. Thus stresses could be one-dimensional, two-dimensional or three-dimensional as shown in the Figure 2.5.
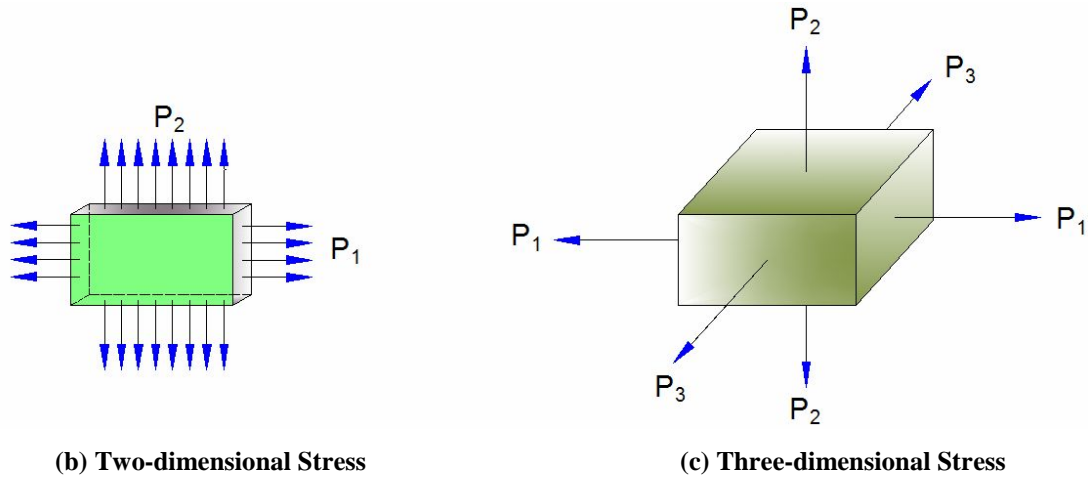
**(a) One-dimensional Stress**
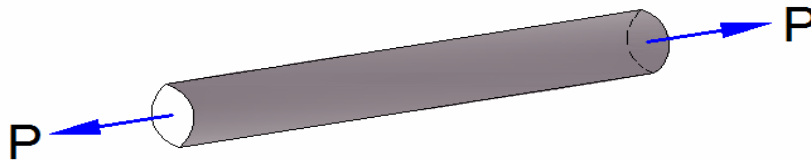
**(b) Two-dimensional Stress**　　　　　　　　**(c) Three-dimensional Stress**

**Figure 2.5 Types of Stress**

## 2.1.8  TWO-DIMENSIONAL STRESS AT A POINT

A two-dimensional state-of-stress exists when the stresses and body forces are independent of one of the co-ordinates. Such a state is described by stresses $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ and the X and Y body forces (Here $z$ is taken as the independent co-ordinate axis).

We shall now determine the equations for transformation of the stress components $\sigma_x$, $\sigma_y$ and $\tau_{xy}$ at any point of a body represented by infinitesimal element as shown in the Figure 2.6.
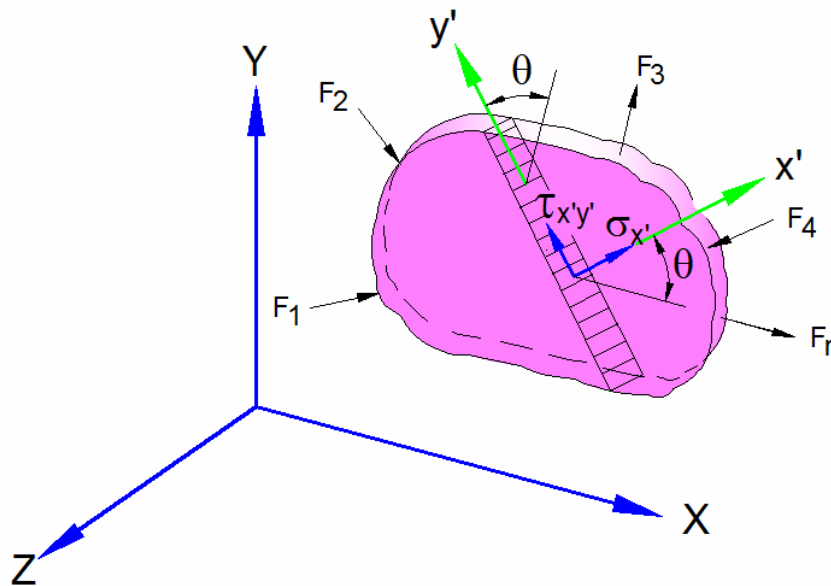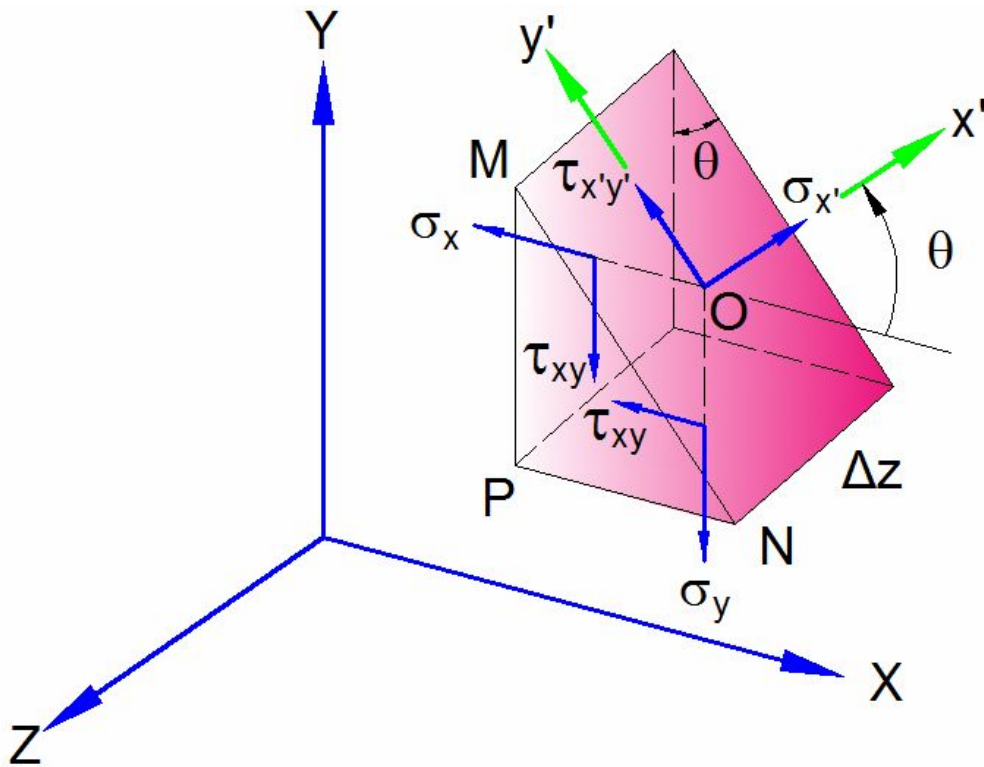


**Figure 2.6 Thin body subjected to stresses in xy plane**

**Figure 2.7 Stress components acting on faces of a small
wedge cut from body of Figure 2.6**

Consider an infinitesimal wedge as shown in Fig.2.7 cut from the loaded body in Figure 2.6. It is required to determine the stresses $\sigma_{x'}$ and $\tau_{x'y'}$, that refer to axes $x'$, $y'$ making an angle $\theta$ with axes $X$, Y as shown in the Figure. Let side MN be normal to the $x'$ axis.

Considering $\sigma_{x'}$ and $\tau_{x'y'}$ as positive and area of side MN as unity, the sides MP and PN have areas $\cos\theta$ and $\sin\theta$, respectively.

Equilibrium of the forces in the $x$ and $y$ directions requires that

$$T_x = \sigma_x \cos\theta + \tau_{xy} \sin\theta$$

$$T_y = \tau_{xy} \cos\theta + \sigma_y \sin\theta \qquad (2.9)$$

where $T_x$ and $T_y$ are the components of stress resultant acting on MN in the $x$ and $y$ directions respectively. The normal and shear stresses on the $x^{'}$ plane (MN plane) are obtained by projecting $T_x$ and $T_y$ in the $x'$ and $y'$ directions.

$$\sigma_{x'} = T_x \cos\theta + T_y \sin\theta \tag{2.10}$$

$$\tau_{x'y'} = T_y \cos\theta - T_x \sin\theta$$

Upon substitution of stress resultants from Equation (2.9), the Equations (2.10) become

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

$$\tau_{x'y'} = \tau_{xy} (\cos^2\theta - \sin^2\theta) + (\sigma_y - \sigma_x) \sin\theta \cos\theta \tag{2.11}$$

The stress $\sigma_{y'}$ is obtained by substituting $\left(\theta + \dfrac{\pi}{2}\right)$ for $\theta$ in the expression for $\sigma_{x'}$.

By means of trigonometric identities

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta), \quad \sin\theta \cos\theta = \frac{1}{2}\sin 2\theta, \tag{2.12}$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

The transformation equations for stresses are now written in the following form:

$$\sigma_{x'} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau_{xy} \sin 2\theta \tag{2.12a}$$

$$\sigma_{y'} = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau_{xy} \sin 2\theta \tag{2.12b}$$

$$\tau_{x'y'} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau_{xy} \cos 2\theta \tag{2.12c}$$

### 2.1.9  PRINCIPAL STRESSES IN TWO DIMENSIONS

To ascertain the orientation of $x'y'$ corresponding to maximum or minimum $\sigma_{x'}$, the necessary condition $\dfrac{d\sigma_{x'}}{d\theta} = 0$, is applied to Equation (2.12a), yielding

$$-(\sigma_x - \sigma_y) \sin 2\theta + 2\tau_{xy} \cos 2\theta = 0 \tag{2.13}$$

Therefore, $\tan 2\theta = \dfrac{2\tau_{xy}}{\sigma_x - \sigma_y}$ $\hspace{4cm}$ (2.14)

As $2\theta = \tan(\pi + 2\theta)$, two directions, mutually perpendicular, are found to satisfy equation (2.14). These are the principal directions, along which the principal or maximum and minimum normal stresses act.

When Equation (2.12c) is compared with Equation (2.13), it becomes clear that $\tau_{x'y'} = 0$ on a principal plane. A principal plane is thus a plane of zero shear. The principal stresses are determined by substituting Equation (2.14) into Equation (2.12a)

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (2.15)$$

Algebraically, larger stress given above is the maximum principal stress, denoted by $\sigma_1$. The minimum principal stress is represented by $\sigma_2$.

Similarly, by using the above approach and employing Equation (2.12c), an expression for the maximum shear stress may also be derived.

## 2.1.10   CAUCHY'S STRESS PRINCIPLE

According to the general theory of stress by Cauchy (1823), the stress principle can be stated as follows:

Consider any closed surface $\partial S$ within a continuum of region B that separates the region B into subregions $B_1$ and $B_2$. The interaction between these subregions can be represented by a field of stress vectors $T(\hat{n})$ defined on $\partial S$. By combining this principle with Euler's equations that expresses balance of linear momentum and moment of momentum in any kind of body, Cauchy derived the following relationship.

$$T\left(\hat{n}\right) = \text{-}T\left(-\hat{n}\right)$$
$$T\left(\hat{n}\right) = \sigma^{\text{T}}\left(\hat{n}\right) \qquad (2.16)$$

where $\left(\hat{n}\right)$ is the unit normal to $\partial S$ and $\sigma$ is the stress matrix. Furthermore, in regions where the field variables have sufficiently smooth variations to allow spatial derivatives upto any order, we have

$$\rho A = div\ \sigma + f \qquad (2.17)$$

where $\rho$ = material mass density

$\quad\quad A$ = acceleration field

$\quad\quad f$ = Body force per unit volume.

This result expresses a necessary and sufficient condition for the balance of linear momentum. When expression (2.17) is satisfied,

$$\sigma = \sigma^{\text{T}} \qquad (2.18)$$

which is equivalent to the balance of moment of momentum with respect to an arbitrary point. In deriving (2.18), it is implied that there are no body couples. If body couples and/or couple stresses are present, Equation (2.18) is modified but Equation (2.17) remains unchanged.

Cauchy Stress principle has four essential ingradients

(i)   The physical dimensions of stress are (force)/(area).

(ii)  Stress is defined on an imaginary surface that separates the region under consideration into two parts.

(iii) Stress is a vector or vector field equipollent to the action of one part of the material on the other.

(iv)  The direction of the stress vector is not restricted.

## 2.1.11   DIRECTION COSINES

Consider a plane ABC having an outward normal *n*. The direction of this normal can be defined in terms of direction cosines. Let the angle of inclinations of the normal with *x*, *y* and *z* axes be $\alpha$, $\beta$ and $\gamma$ respectively. Let $P(x, y, z)$ be a point on the normal at a radial distance *r* from the origin *O*.
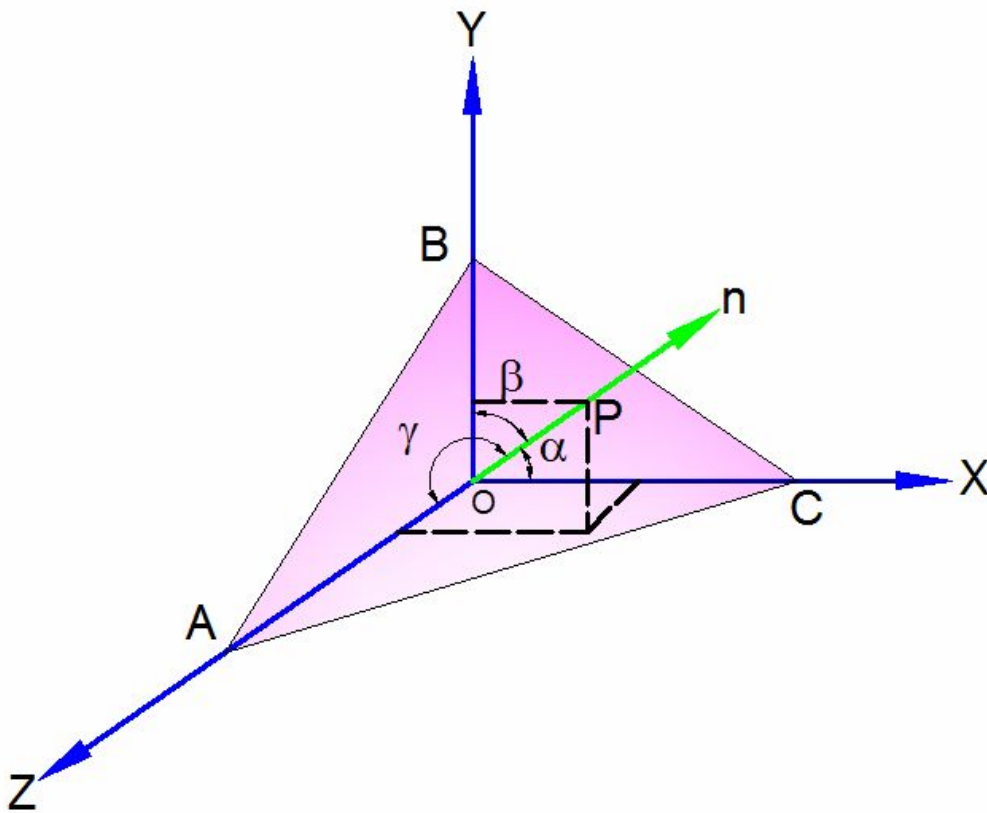


**Figure 2.8 Tetrahedron with arbitrary plane**

From figure,

$$\cos\alpha = \frac{x}{r}, \quad \cos\beta = \frac{y}{r} \text{ and } \cos\gamma = \frac{z}{r}$$

or $\quad x = r\cos\alpha, \quad y = r\cos\beta \text{ and } z = r\cos\gamma$

Let $\quad \cos\alpha = l, \quad \cos\beta = m \text{ and } \cos\gamma = n$

Therefore, $\dfrac{x}{r} = l, \quad \dfrac{y}{r} = m \text{ and } \dfrac{z}{r} = n$

Here, *l*, *m* and *n* are known as direction cosines of the line *OP*. Also, it can be written as

$$x^2 + y^2 + z^2 = r^2 \text{ (since } r \text{ is the polar co-ordinate of } P)$$

or $\quad \dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} + \dfrac{z^2}{r^2} = 1$

$$l^2 + m^2 + n^2 = 1$$

## 2.1.12 STRESS COMPONENTS ON AN ARBITRARY PLANE

Consider a small tetrahedron isolated from a continuous medium (Figure 2.9) subjected to a general state of stress. The body forces are taken to be negligible. Let the arbitrary plane *ABC* be identified by its outward normal *n* whose direction cosines are *l*, *m* and *n*.

In the Figure 2.9, $T_x, T_y, T_z$ are the Cartesian components of stress resultant *T*, acting on oblique plane ABC. It is required to relate the stresses on the perpendicular planes intersecting at the origin to the normal and shear stresses acting on ABC.

The orientation of the plane ABC may be defined in terms of the angle between a unit normal *n* to the plane and the *x, y, z* directions. The direction cosines associated with these angles are

cos (*n, x*) = *l*
cos (*n, y*) = *m* $\qquad$ and $\hspace{5cm}$ (2.19)
cos (*n, z*) = *n*

The three direction cosines for the *n* direction are related by

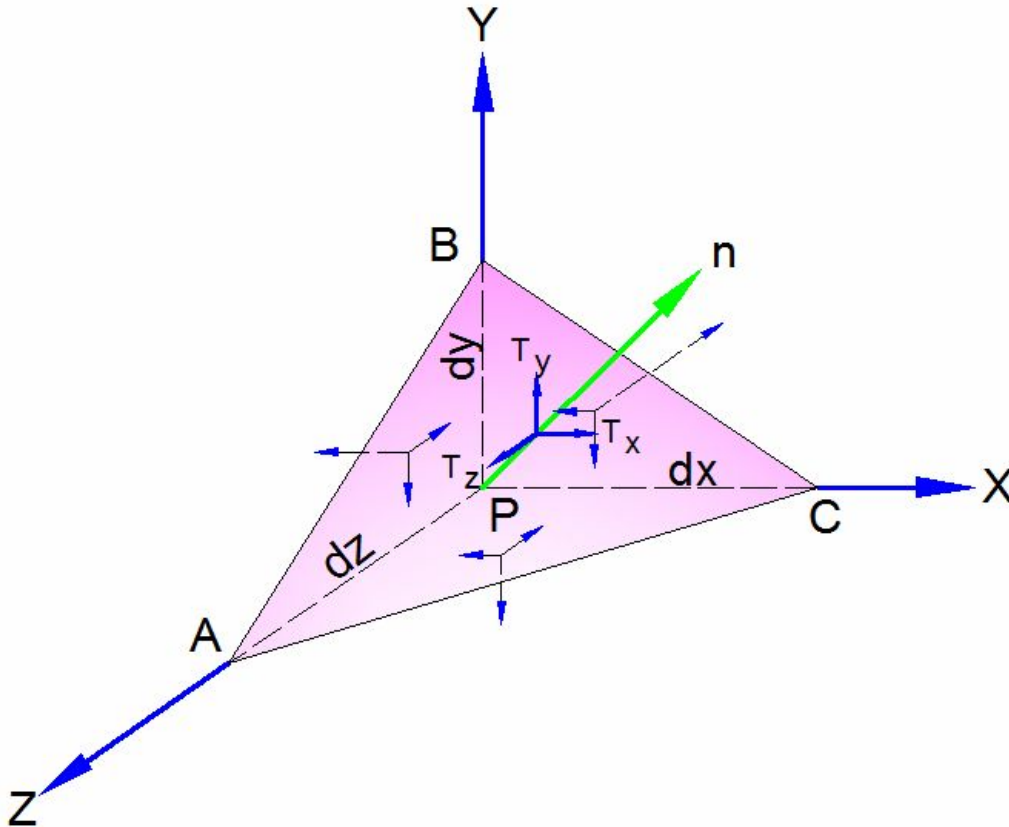$l^2 + m^2 + n^2 = 1$ $\hspace{8cm}$ (2.20)

**Figure 2.9 Stresses acting on face of the tetrahedron**

The area of the perpendicular plane PAB, PAC, PBC may now be expressed in terms of *A*, the area of ABC, and the direction cosines.

Therefore, Area of PAB $= A_{PAB} = A_x = A.i$

$$= A \ (li + mj + nk) \ i$$

Hence,  $A_{\text{PAB}} = Al$

The other two areas are similarly obtained.  In doing so, we have altogether

$A_{\text{PAB}} = Al, A_{\text{PAC}} = Am, A_{\text{PBC}} = An$ (2.21)

Here *i, j* and *k* are unit vectors in *x, y* and *z* directions, respectively.

Now, for equilibrium of the tetrahedron, the sum of forces in *x, y* and *z* directions must be zero.

Therefore, $T_x A = \sigma_x Al + \tau_{xy} Am + \tau_{xz} An$ (2.22)

Dividing throughout by *A*, we get

$$T_x = \sigma_x\, l + \tau_{xy}\, m + \tau_{xz}\, n \tag{2.22a}$$

Similarly, for equilibrium in *y* and *z* directions,

$$T_y = \tau_{xy}\, l + \sigma_y\, m + \tau_{yz}\, n \tag{2.22b}$$

$$T_z = \tau_{xz}\, l + \tau_{yz}\, m + \sigma_z\, n \tag{2.22c}$$

The stress resultant on *A* is thus determined on the basis of known stresses $\sigma_x, \sigma_y, \sigma_z,$ $\tau_{xy}, \tau_{yz}, \tau_{zx}$ and a knowledge of the orientation of *A*.

The Equations (2.22a), (2.22b) and (2.22c) are known as Cauchy's stress formula. These equations show that the nine rectangular stress components at *P* will enable one to determine the stress components on any arbitrary plane passing through point *P*.

## 2.1.13 STRESS TRANSFORMATION

When the state or stress at a point is specified in terms of the six components with reference to a given co-ordinate system, then for the same point, the stress components with reference to another co-ordinate system obtained by rotating the original axes can be determined using the direction cosines.

Consider a cartesian co-ordinate system X, Y and Z as shown in the Figure 2.10. Let this given co-ordinate system be rotated to a new co-ordinate system $\mathrm{x}', \mathrm{y}', \mathrm{z}'$ where in $x'$ lie on an oblique plane. $\mathrm{x}', \mathrm{y}', \mathrm{z}'$ and X, Y, Z systems are related by the direction cosines.

$$l_1 = \cos(\mathrm{x}', X)$$
$$m_1 = \cos(\mathrm{x}', Y) \tag{2.23}$$
$$n_1 = \cos(\mathrm{x}', Z)$$

(The notation corresponding to a complete set of direction cosines is shown in Table 1.0).

**Table 1.0 Direction cosines relating different axes**

|        | *X*   | *Y*   | *Z*   |
|--------|-------|-------|-------|
| x'     | $l_1$ | $m_1$ | $n_1$ |
| y'     | $l_2$ | $m_2$ | $n_2$ |
| z'     | $l_3$ | $m_3$ | $n_3$ |
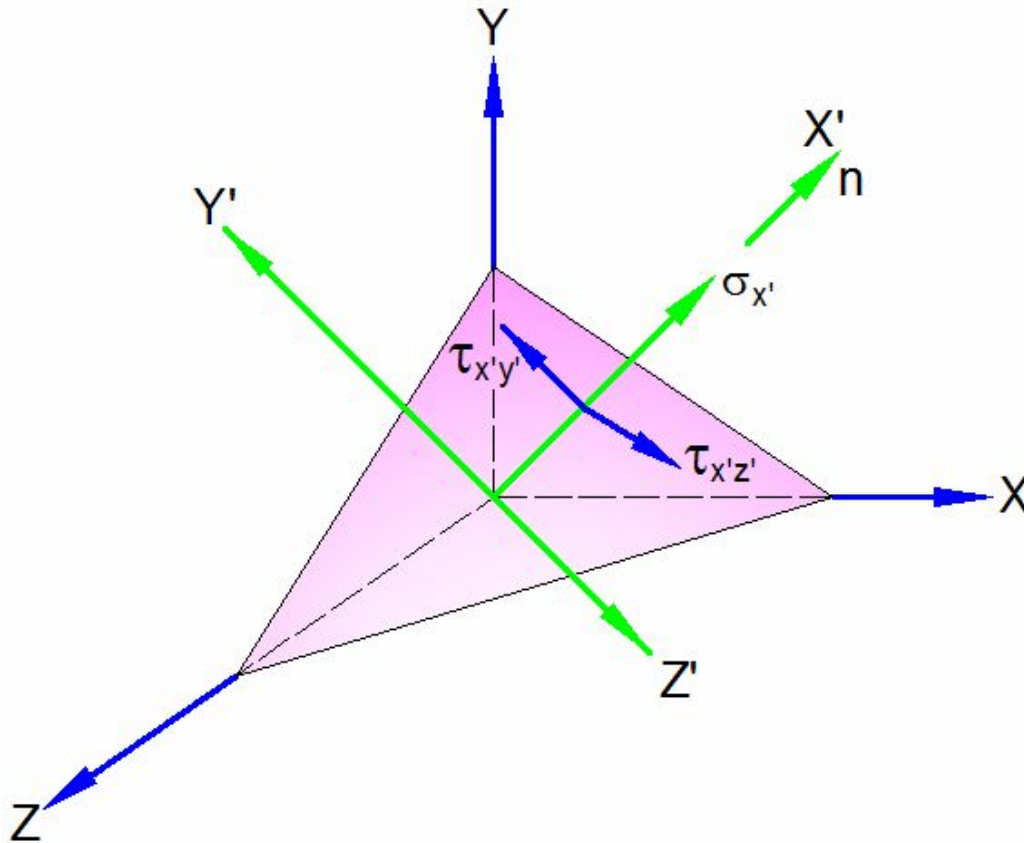
**Figure 2.10 Transformation of co-ordinates**

The normal stress $\sigma_{x'}$ is found by projecting $T_x, T_y$ and $T_z$ in the $\mathrm{x}'$ direction and adding:

$$\sigma_{x'} = T_x \, l_1 + T_y \, m_1 + T_z \, n_1 \tag{2.24}$$

Equations (2.22a), (2.22b), (2.22c) and (2.24) are combined to yield

$$\sigma_{x'} = \sigma_x \, l_1^2 + \sigma_y \, m_1^2 + \sigma_z \, n_1^2 + 2(\tau_{xy} \, l_1 \, m_1 + \tau_{yz} \, m_1 \, n_1 + \tau_{xz} \, l_1 \, n_1) \tag{2.25}$$

Similarly by projecting $T_x, T_y, T_z$ in the $y'$ and $z'$ directions, we obtain, respectively

$$\tau_{x'y'} = \sigma_x \, l_1 \, l_2 + \sigma_y \, m_1 \, m_2 + \sigma_z \, n_1 \, n_2 + \tau_{xy} \, (l_1 \, m_2 + m_1 \, l_2) + \tau_{yz} \, (m_1 \, n_2 + n_1 \, m_2) + \tau_{xz} \, (n_1 l_2 + l_1 n_2)$$
$$\tag{2.25a}$$

$$\tau_{x'z'} = \sigma_x \, l_1 \, l_3 + \sigma_y \, m_1 \, m_3 + \sigma_z \, n_1 \, n_3 + \tau_{xy} \, (l_1 \, m_3 + m_1 \, l_3) + \tau_{yz} \, (m_1 \, n_3 + n_1 \, m_3) + \tau_{xz} \, (n_1 \, l_3 + l_1 \, n_3)$$
$$\tag{2.25b}$$

Recalling that the stresses on three mutually perpendicular planes are required to specify the stress at a point (one of these planes being the oblique plane in question), the remaining components are found by considering those planes perpendicular to the oblique plane. For one such plane *n* would now coincide with $y'$ direction, and expressions for the stresses

$\sigma_{y'}, \tau_{y'}, \tau_{y'z'}$ would be derived. In a similar manner the stresses $\sigma_{z'}, \tau_{z'x'}, \tau_{z'y'}$ are determined when $n$ coincides with the $z'$ direction. Owing to the symmetry of stress tensor, only six of the nine stress components thus developed are unique. The remaining stress components are as follows:

$$\sigma_{y'} = \sigma_x l_2^2 + \sigma_y m_2^2 + \sigma_z n_2^2 + 2 \left( \tau_{xy} l_2 m_2 + \tau_{yz} m_2 n_2 + \tau_{xz} l_2 n_2 \right) \tag{2.25c}$$

$$\sigma_{z'} = \sigma_x l_3^2 + \sigma_y m_3^2 + \sigma_z n_3^2 + 2 \left( \tau_{xy} l_3 m_3 + \tau_{yz} m_3 n_3 + \tau_{xz} l_3 n_3 \right) \tag{2.25d}$$

$$\tau_{y'z'} = \sigma_x l_2 l_3 + \sigma_y m_2 m_3 + \sigma_z n_2 n_3 + \tau_{xy} (m_2 l_3 + l_2 m_3) + \tau_{yz} (n_2 m_3 + m_2 n_3) + \tau_{xz} (l_2 n_3 + n_2 l_3)$$
$$\tag{2.25e}$$

The Equations (2.25 to 2.25e) represent expressions transforming the quantities $\sigma_x, \sigma_y, \tau_{xy}, \tau_{yz}, \tau_{xz}$ to completely define the state of stress.

It is to be noted that, because $x'$, $y'$ and $z'$ are orthogonal, the nine direction cosines must satisfy trigonometric relations of the following form.

$$l_i^2 + m_i^2 + n_i^2 = 1 \qquad (i = 1,2,3)$$

and
$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$
$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \tag{2.26}$$
$$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0$$

*Applied Elasticity for Engineers*                    *T .G.Sitharam & L.GovindaRaju*