Unsymmetric Bending

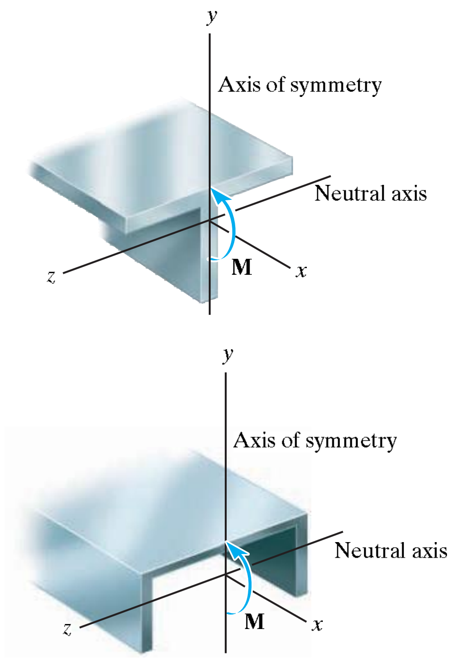


Fig. 6–29

When developing the flexure formula we imposed a condition that the cross-sectional area be *symmetric* about an axis perpendicular to the neutral axis; furthermore, the resultant internal moment **M** acts along the neutral axis. Such is the case for the "T" or channel sections shown in Fig. 6–29. These conditions, however, are unnecessary, and in this section we will show that the flexure formula can also be applied either to a beam having a cross-sectional area of any shape or to a beam having a resultant internal moment that acts in any direction.

Moment Applied About Principal Axis. Consider the beam's cross section to have the unsymmetrical shape shown in Fig. 6–30*a*. As in Sec. 6.4, the right-handed x, y, z coordinate system is established such that the origin is located at the centroid C on the cross section, and the resultant internal moment **M** acts along the $+z$ axis. We require the stress distribution acting over the entire cross-sectional area to have a zero force resultant, the resultant internal moment about the y axis to be zero, and the resultant internal moment about the z axis to equal **M**.* These three conditions can be expressed mathematically by considering the force acting on the differential element dA located at $(0, y, z)$, Fig. 6–30*a*. This force is $dF = \sigma\, dA$, and therefore we have

$$F_R = \Sigma F_x; \qquad 0 = -\int_A \sigma\, dA \tag{6–14}$$

$$(M_R)_y = \Sigma M_y; \qquad 0 = -\int_A z\sigma\, dA \tag{6–15}$$

$$(M_R)_z = \Sigma M_z; \qquad M = \int_A y\sigma\, dA \tag{6–16}$$

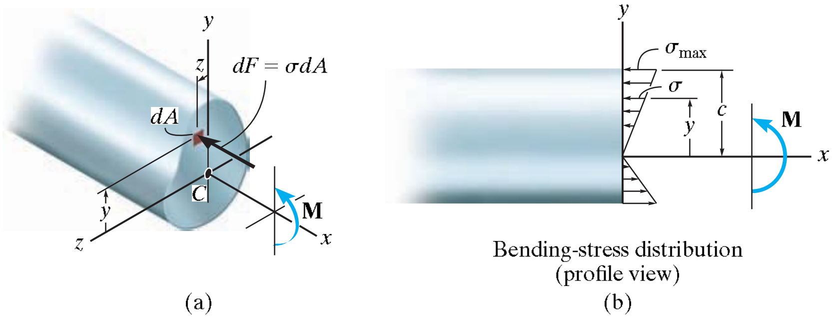


Fig. 6–30

*The condition that moments about the y axis be equal to zero was not considered in Sec. 6.4, since the bending-stress distribution was *symmetric* with respect to the y axis and such a distribution of stress automatically produces zero moment about the y axis. See Fig. 6–24*c*.

As shown in Sec. 6.4, Eq. 6–14 is satisfied since the z axis passes through the *centroid* of the area. Also, since the z axis represents the *neutral axis* for the cross section, the normal stress will vary linearly from zero at the neutral axis, to a maximum at $y = c$, Fig. 6–30*b*. Hence the stress distribution is defined by $\sigma = -(y/c)\sigma_{\max}$. When this equation is substituted into Eq. 6–16 and integrated, it leads to the flexure formula $\sigma_{\max} = Mc/I$. When it is substituted into Eq. 6–15, we get

$$0 = \frac{-\sigma_{\max}}{c} \int_A yz\, dA$$

which requires

$$\int_A yz\, dA = 0$$

This integral is called the ***product of inertia*** for the area. As indicated in Appendix A, it will indeed be zero provided the y and z axes are chosen as ***principal axes of inertia*** for the area. For an arbitrarily shaped area, the orientation of the principal axes can always be determined, using either the inertia transformation equations or Mohr's circle of inertia as explained in Appendix A, Secs. A.4 and A.5. If the area has an axis of symmetry, however, the ***principal axes*** can easily be established ***since they will always be oriented along the axis of symmetry and perpendicular to it***.

For example, consider the members shown in Fig. 6–31. In each of these cases, y and z must define the principal axes of inertia for the cross section in order to satisfy Eqs. 6–14 through 6–16. In Fig. 6–31*a* the principal axes are located by symmetry, and in Figs. 6–31*b* and 6–31*c* their orientation is determined using the methods of Appendix A. Since **M** is applied about one of the principal axes (z axis), the stress distribution is determined from the flexure formula, $\sigma = -My/I_z$, and is shown for each case.

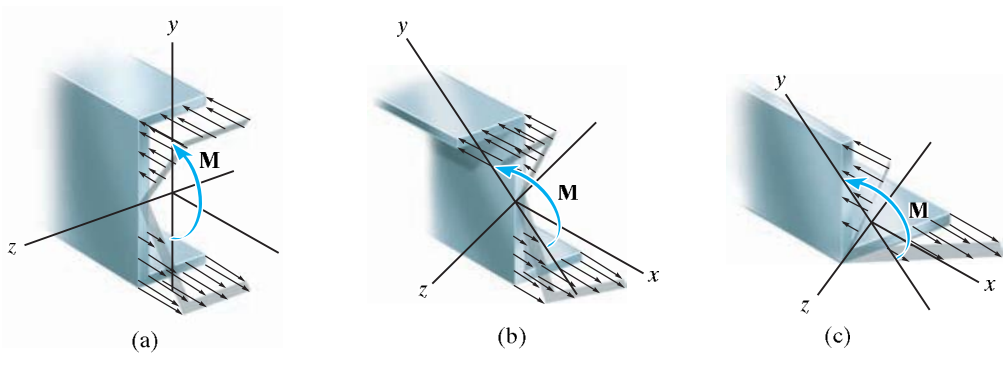


Fig. 6–31

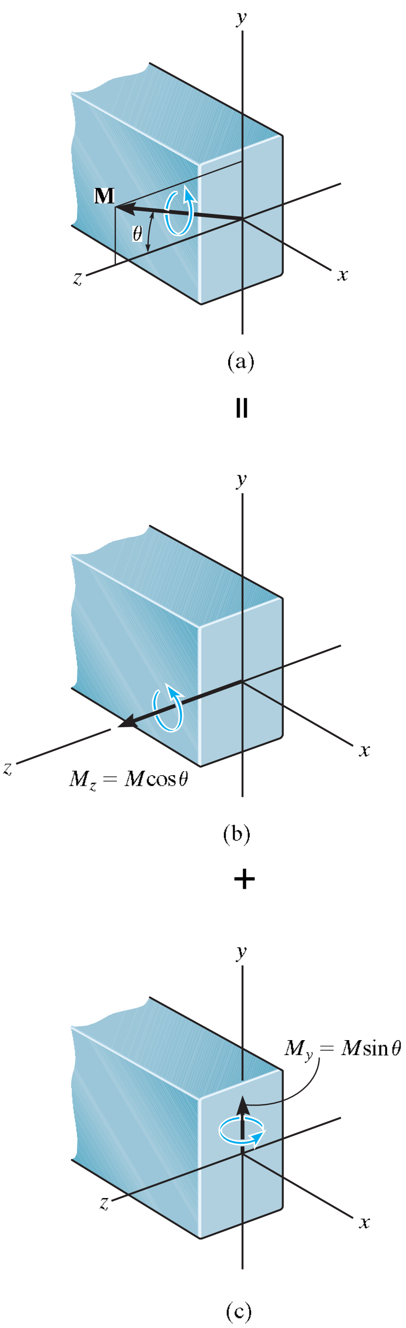


Fig. 6–32

Moment Arbitrarily Applied. Sometimes a member may be loaded such that **M** does not act about one of the principal axes of the cross section. When this occurs, the moment should first be resolved into components directed along the principal axes, then the flexure formula can be used to determine the normal stress caused by *each* moment component. Finally, using the principle of superposition, the resultant normal stress at the point can be determined.

To show this, consider the beam to have a rectangular cross section and to be subjected to the moment **M**, Fig. 6–32*a*. Here **M** makes an angle θ with the *principal* z axis. We will assume θ is positive when it is directed from the $+z$ axis toward the $+y$ axis, as shown. Resolving **M** into components along the z and y axes, we have $M_z = M\cos\theta$ and $M_y = M\sin\theta$, as shown in Figs. 6–32*b* and 6–32*c*. The normal-stress distributions that produce **M** and its components $\mathbf{M}_z$ and $\mathbf{M}_y$ are shown in Figs. 6–32*d*, 6–32*e*, and 6–32*f*, where it is assumed that $(\sigma_x)_{\max} > (\sigma'_x)_{\max}$. By inspection, the maximum tensile and compressive stresses $[(\sigma_x)_{\max} + (\sigma'_x)_{\max}]$ occur at two opposite corners of the cross section, Fig. 6–32*d*.

Applying the flexure formula to each moment component in Figs. 6–32*b* and 6–32*c*, and adding the results algebraically, the resultant normal stress at any point on the cross section, Fig. 6–32*d*, is then

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \qquad (6\text{–}17)$$

Here,

σ = the normal stress at the point

y, z = the coordinates of the point measured from x, y, z axes having their origin at the centroid of the cross-sectional area and forming a right-handed coordinate system The x axis is directed outward from the cross-section and the y and z axes represent respectively the principal axes of minimum and maximum moment of inertia for the area

M_y, M_z = the resultant internal moment components directed along the principal y and z axes. They are positive if directed along the $+y$ and $+z$ axes, otherwise they are negative. Or, stated another way, $M_y = M\sin\theta$ and $M_z = M\cos\theta$, where θ is measured positive from the $+z$ axis toward the $+y$ axis

I_y, I_z = the *principal moments of inertia* calculated about the y and z axes, respectively. See Appendix A

The x, y, z axes form a right-handed system, and the proper algebraic signs must be assigned to the moment components and the coordinates when applying this equation. When this is the case, the resulting stress will be *tensile* if it is *positive* and *compressive* if it is *negative*.

Orientation of the Neutral Axis. The angle α of the neutral axis in Fig. 6–32*d* can be determined by applying Eq. 6–17 with $\sigma = 0$, since by definition no normal stress acts on the neutral axis. We have

$$y = \frac{M_y I_z}{M_z I_y} z$$

Since $M_z = M \cos\theta$ and $M_y = M \sin\theta$, then

$$y = \left(\frac{I_z}{I_y}\tan\theta\right) z \qquad (6\text{–}18)$$

This equation defines the neutral axis for the cross section. Since the slope of this line is $\tan\alpha = y/z$, then

$$\tan\alpha = \frac{I_z}{I_y}\tan\theta \qquad (6\text{–}19)$$

Here it can be seen that unless $I_z = I_y$ the angle θ, defining the direction of the moment **M**, Fig. 6–32*a*, will *not equal* α, the angle defining the inclination of the neutral axis, Fig. 6–32*d*.

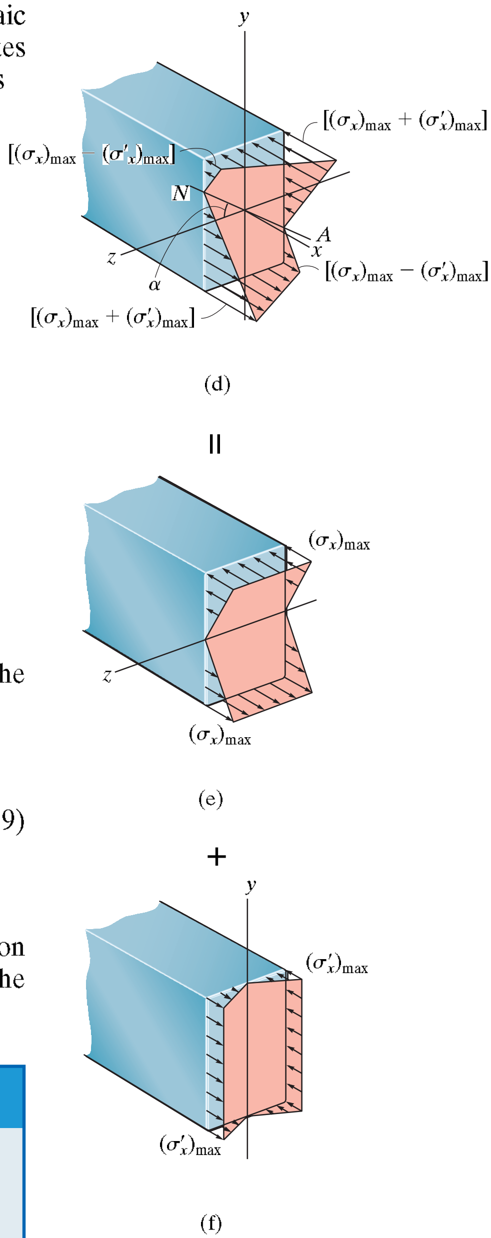


Fig. 6–32 (cont.)

Important Points

- The flexure formula can be applied only when bending occurs about axes that represent the *principal axes of inertia* for the cross section. These axes have their origin at the centroid and are oriented along an axis of symmetry, if there is one, and perpendicular to it.
- If the moment is applied about some arbitrary axis, then the moment must be resolved into components along each of the principal axes, and the stress at a point is determined by superposition of the stress caused by each of the moment components.

EXAMPLE 6.15

The rectangular cross section shown in Fig. 6–33*a* is subjected to a bending moment of $M = 12\ \text{kN}\cdot\text{m}$. Determine the normal stress developed at each corner of the section, and specify the orientation of the neutral axis.

SOLUTION

Internal Moment Components. By inspection it is seen that the y and z axes represent the principal axes of inertia since they are axes of symmetry for the cross section. As required we have established the z *axis* as the principal axis for *maximum* moment of inertia. The moment is resolved into its y and z components, where

$$M_y = -\frac{4}{5}(12\ \text{kN}\cdot\text{m}) = -9.60\ \text{kN}\cdot\text{m}$$

$$M_z = \frac{3}{5}(12\ \text{kN}\cdot\text{m}) = 7.20\ \text{kN}\cdot\text{m}$$

Section Properties. The moments of inertia about the y and z axes are

$$I_y = \frac{1}{12}(0.4\ \text{m})(0.2\ \text{m})^3 = 0.2667(10^{-3})\ \text{m}^4$$

$$I_z = \frac{1}{12}(0.2\ \text{m})(0.4\ \text{m})^3 = 1.067(10^{-3})\ \text{m}^4$$

Bending Stress. Thus,

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

$$\sigma_B = -\frac{7.20(10^3)\ \text{N}\cdot\text{m}(0.2\ \text{m})}{1.067(10^{-3})\ \text{m}^4} + \frac{-9.60(10^3)\ \text{N}\cdot\text{m}(-0.1\ \text{m})}{0.2667(10^{-3})\ \text{m}^4} = 2.25\ \text{MPa} \qquad \textit{Ans.}$$

$$\sigma_C = -\frac{7.20(10^3)\ \text{N}\cdot\text{m}(0.2\ \text{m})}{1.067(10^{-3})\ \text{m}^4} + \frac{-9.60(10^3)\ \text{N}\cdot\text{m}(0.1\ \text{m})}{0.2667(10^{-3})\ \text{m}^4} = -4.95\ \text{MPa} \qquad \textit{Ans.}$$

$$\sigma_D = -\frac{7.20(10^3)\ \text{N}\cdot\text{m}(-0.2\ \text{m})}{1.067(10^{-3})\ \text{m}^4} + \frac{-9.60(10^3)\ \text{N}\cdot\text{m}(0.1\ \text{m})}{0.2667(10^{-3})\ \text{m}^4} = -2.25\ \text{MPa} \qquad \textit{Ans.}$$

$$\sigma_E = -\frac{7.20(10^3)\ \text{N}\cdot\text{m}(-0.2\ \text{m})}{1.067(10^{-3})\ \text{m}^4} + \frac{-9.60(10^3)\ \text{N}\cdot\text{m}(-0.1\ \text{m})}{0.2667(10^{-3})\ \text{m}^4} = 4.95\ \text{MPa} \qquad \textit{Ans.}$$

The resultant normal-stress distribution has been sketched using these values, Fig. 6–33*b*. Since superposition applies, the distribution is linear as shown.

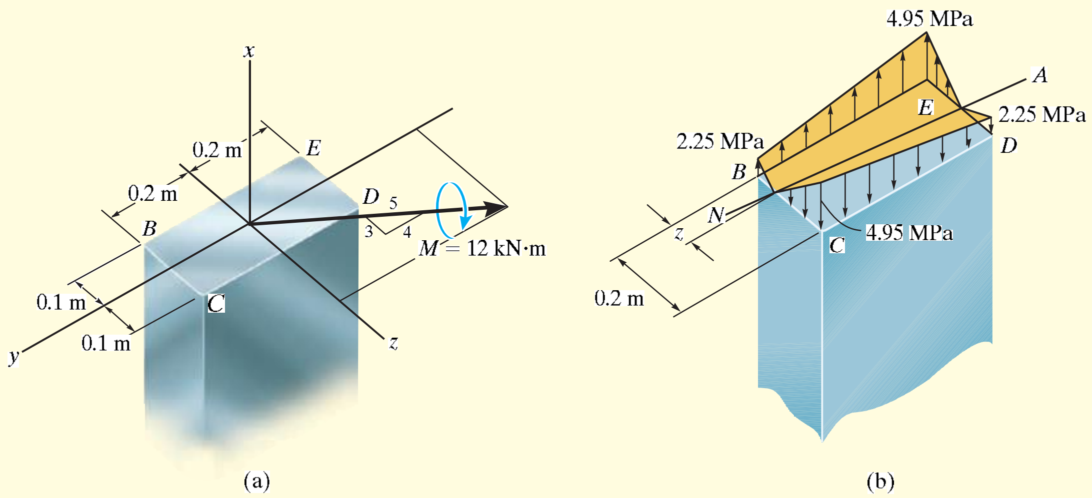


Fig. 6–33

Orientation of Neutral Axis. The location z of the neutral axis (NA), Fig. 6–33b, can be established by proportion. Along the edge BC, we require

$$\frac{2.25\text{ MPa}}{z} = \frac{4.95\text{ MPa}}{(0.2\text{ m} - z)}$$

$$0.450 - 2.25z = 4.95z$$

$$z = 0.0625\text{ m}$$

In the same manner this is also the distance from D to the neutral axis in Fig. 6–33b.

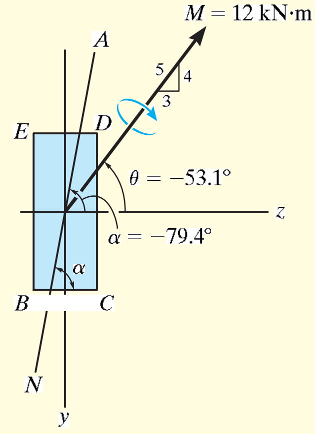
We can also establish the orientation of the NA using Eq. 6–19, which is used to specify the angle α that the axis makes with the z or *maximum* principal axis. According to our sign convention, θ must be measured from the $+z$ axis toward the $+y$ axis. By comparison, in Fig. 6–33c, $\theta = -\tan^{-1}\frac{4}{3} = -53.1^\circ$ (or $\theta = +306.9^\circ$). Thus,

$$\tan\alpha = \frac{I_z}{I_y}\tan\theta$$

$$\tan\alpha = \frac{1.067(10^{-3})\text{ m}^4}{0.2667(10^{-3})\text{ m}^4}\tan(-53.1^\circ)$$

$$\alpha = -79.4^\circ \qquad \textit{Ans.}$$

This result is shown in Fig. 6–33c. Using the value of z calculated above, verify, using the geometry of the cross section, that one obtains the same answer.



(c)

EXAMPLE 6.16

The Z-section shown in Fig. 6–34a is subjected to the bending moment of $M = 20\ \text{kN}\cdot\text{m}$. Using the methods of Appendix A (see Example A.4 or A.5), the principal axes y and z are oriented as shown, such that they represent the minimum and maximum principal moments of inertia, $I_y = 0.960(10^{-3})\ \text{m}^4$ and $I_z = 7.54(10^{-3})\ \text{m}^4$, respectively. Determine the normal stress at point P and the orientation of the neutral axis.

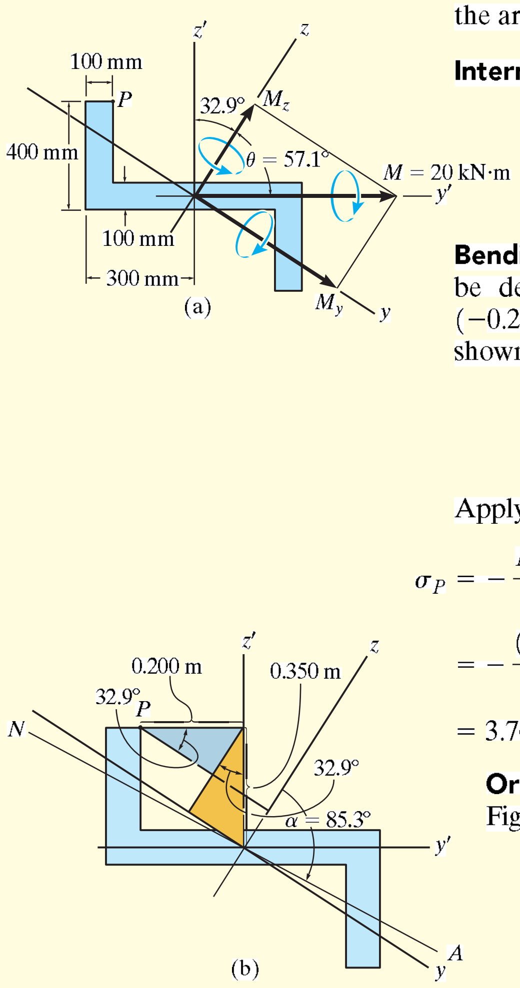


Fig. 6–34

SOLUTION

For use of Eq. 6–19, it is important that the z axis represent the principal axis for the *maximum* moment of inertia. (Note that most of the area is located furthest from this axis.)

Internal Moment Components. From Fig. 6–34a,

$$M_y = 20\ \text{kN}\cdot\text{m}\sin 57.1^\circ = 16.79\ \text{kN}\cdot\text{m}$$

$$M_z = 20\ \text{kN}\cdot\text{m}\cos 57.1^\circ = 10.86\ \text{kN}\cdot\text{m}$$

Bending Stress. The y and z coordinates of point P must be determined first. Note that the y', z' coordinates of P are $(-0.2\ \text{m}, 0.35\ \text{m})$. Using the colored triangles from the construction shown in Fig. 6–34b, we have

$$y_P = -0.35\sin 32.9^\circ - 0.2\cos 32.9^\circ = -0.3580\ \text{m}$$

$$z_P = 0.35\cos 32.9^\circ - 0.2\sin 32.9^\circ = 0.1852\ \text{m}$$

Applying Eq. 6–17,

$$\sigma_P = -\frac{M_z y_P}{I_z} + \frac{M_y z_P}{I_y}$$

$$= -\frac{(10.86(10^3)\ \text{N}\cdot\text{m})(-0.3580\ \text{m})}{7.54(10^{-3})\ \text{m}^4} + \frac{(16.79(10^3)\ \text{N}\cdot\text{m})(0.1852\ \text{m})}{0.960(10^{-3})\ \text{m}^4}$$

$= 3.76\ \text{MPa}$ *Ans.*

Orientation of Neutral Axis. The angle $\theta = 57.1^\circ$ is shown in Fig. 6–34a. Thus,

$$\tan\alpha = \left[\frac{7.54(10^{-3})\ \text{m}^4}{0.960(10^{-3})\ \text{m}^4}\right]\tan 57.1^\circ$$

$\alpha = 85.3^\circ$ *Ans.*

The neutral axis is oriented as shown in Fig. 6–34b.

UNSYMMETRICAL BENDING

Unsymmetrical bending involves the cases in which either the bending moments do not act in a plane of symmetry of the member or the member does not possess a symmetric cross-sectional area.

In deriving the relation for pure bending, one of the assumptions is that the section is symmetrical about a vertical plane passing through the vertical axes of symmetry and the bending couple acts in that plane. A vertical axis of symmetry is perpendicular to the neutral axis passing through the centroid. Owing to symmetry of such a member and of the loading, the member remains symmetric with respect to the vertical plane and is bent in that plane. If the vertical plane is not a plane of symmetry, the member cannot be expected to bend in that plane.

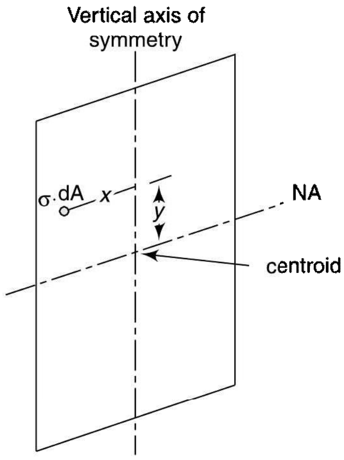


Fig. 5.47

For symmetric bending, the summation of moments of all the elementary forces about the vertical axis of symmetry must be zero, i.e.,

$$\int \sigma dA \cdot x = 0 \quad \text{or} \quad \int \sigma x\, dA \cdot = 0$$

or $$\frac{E}{R}\int y\, x\, dA = 0 \qquad \left(\because \frac{\sigma}{y} = \frac{E}{R}\right)$$

or $$\int xydA = 0 \tag{5.18}$$

which is the necessary condition to use the bending equation derived earlier. However, this condition is found to be satisfied for a set of two perpendicular axes for all types of symmetric and unsymmetric sections. The integral $\int xydA$ (also denoted by I_{xy}) is known as *product of inertia.* The two axes for which it is zero for a section are known as *principal axes* or *principlal centroidal axes* of the cross-section. The moments of inertia of an area about its principal axes are known as *principal moments of inertia.* If a cross-section has an axis of symmetry, then it can easily be shown that this satisfies the condition for a principal axis. The other principal axis will be at right angle through the centroid.

- Note that the product of inertia $I_{xy} = \int xydA$ is not the product of $I_x = \int y^2dA$ and $I_y = \int x^2dA$. Thus whereas I_{xy} is zero for principal axes, I_x and I_y will have certain values known as *principal moments of inertia.*
- In case of unsymmetrical bending it is assumed that there is no twisting of the members due to unsymmetric shear stresses. It is observed that if the load is applied through a particular point known as *shear centre*, there will not be any torsion or twisting of a member due to shear stresses. The shear centre may lie in or outside the section. If the load is not applied through the shear centre, there is twisting of the beam due to unbalanced moment caused by the shear force acting on the section. For sections symmetrical about an axis, the shear centre lies on the axis of symmetry. For sections having two axes of symmetry, the shear centre lies at the intersection of these axes and thus coincides with the centroid. (Refer Section 6.5 for details).

Example 5.34 A 4-m long simply supported beam of 80 mm width and 100 mm depth carries a load of 10 kN at the midspan. The load is inclined at 30° to the vertical longitudinal plane and the line of action of the load passes through the centroid of the rectangular section of the beam. Determine the stresses at all the corners of the section.

Solution

Given A simply supported beam carrying an inclined load as shown in Fig. 5.48a

$L = 4$ m $\quad W = 10$ kN

To find Stresses at all corners of the section

The section being symmetrical, the centroid is at the centre of the rectangle and the coordinate axes *x-x* and *y-y* are also the principal axes (Fig. 5.48).

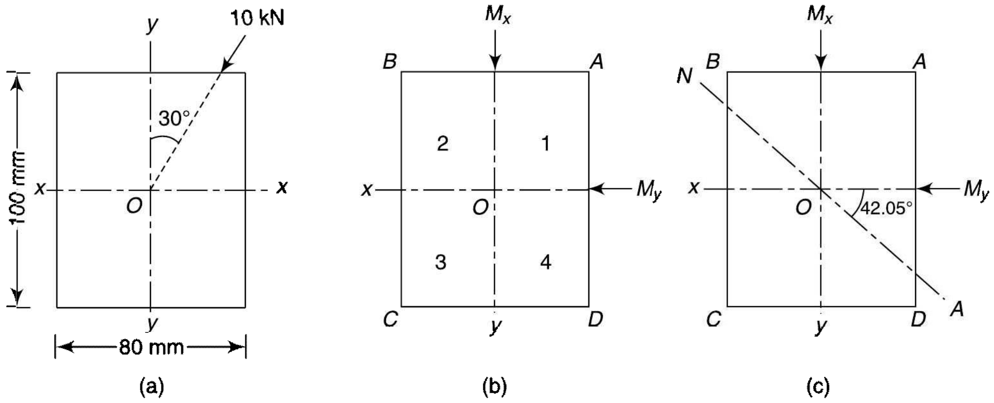


Fig. 5.48

Moment of inertia

$$I_x = \frac{80 \times 100^3}{12} = 6.667 \times 10^6 \text{ mm}^4$$

and
$$I_y = \frac{100 \times 80^3}{12} = 4.267 \times 10^6 \text{ mm}^4$$

Bending moments

Maximum bending moment $= \dfrac{10 \times 4}{4} = 10$ kN·m or 10×10^6 N·mm

Resolving into components,

$M_x = 10 \times 10^6 \cos 30° = 8.66 \times 10^6$ N·mm (due to vertical component of load)

$M_y = 10 \times 10^6 \sin 30° = 5 \times 10^6$ N·mm (due to horizontal component of load)

As it is a simply supported beam,

- the vertical load component induces compressive stress in the upper half and tensile stress in the lower half, and
- the horizontal load component induces compressive stress in the right half and tensile stress in the left half.

Bending stresses

The bending stress at any point (x, y) in the section consists of two parts, one due to bending about the axis *x-x* and the other due to the bending about *y-y*, i.e.,

$$\sigma = \frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x$$

As both the components are to give tensile stress in the 3rd quadrant, x and y both can be assumed positive in this quadrant. Thus assume the following for positive and negative directions for x and y:

- x positive towards left of O and negative towards right of O
- y positive downward from O and negative upwards from O

Thus

— in 1st , x and y both negative

— in 2nd, x positive and y negative

— in 3rd , x and y both positive and

— in 4th , x negative and y positive

Therefore,

$$\sigma_a = \frac{8.66 \times 10^6}{6.667 \times 10^6} \cdot (-40) + \frac{5 \times 10^6}{4.267 \times 10^6} \cdot (-50)$$

$$= -1.299 \times 40 - 1.172 \times 50 = -111.83 \text{ MPa}$$

$$\sigma_b = 1.299 \times 40 + 1.172 \times (-50) = 64.95 - 46.88 = -18.07 \text{ MPa}$$

$$\sigma_c = 1.299 \times 40 + 1.172 \times 50 = 64.95 + 46.88 = 111.83 \text{ MPa}$$

$$\sigma_d = 1.299 \times (-40) + 1.172 \times 50 = -64.95 + 46.88 = 18.07 \text{ MPa}$$

Neutral axis

As the stress at the neutral axis is zero, the equation for the neutral axis is $\sigma = 0$

or $\quad 1.299\, y + 1.172 \times = 0$

or $\quad \tan\alpha = \frac{y}{x} = -\frac{1.172}{1.299} = -0.902 \quad$ or $\quad \alpha = -42.05°$

As it is negative, it is taken at an angle of 42.05° with the x-axis in such a way that it passes through a quadrant in which either x or y is negative. The neutral axis has been shown in the figure. Finally, on that side of the neutral axis which has 3rd quadrant (both x and y positive), there will be tensile stress and on the other compressive stress.

Example 5.35 ‖ A simply supported I-beam of 2-m span carries a central load of 4 kN. The load acts through the centroid, the line of action is inclined at 30° to the vertical direction. Determine the maximum stress.

Solution

Given A simply supported I-beam carrying an inclined load as shown in Fig. 5.49

$L = 2$ m $\qquad W = 4$ kN

To find Maximum stress

As the section is symmetrical, the centroid is at the centre of the web and x-x and the coordinate axes y-y are also the principal axes.

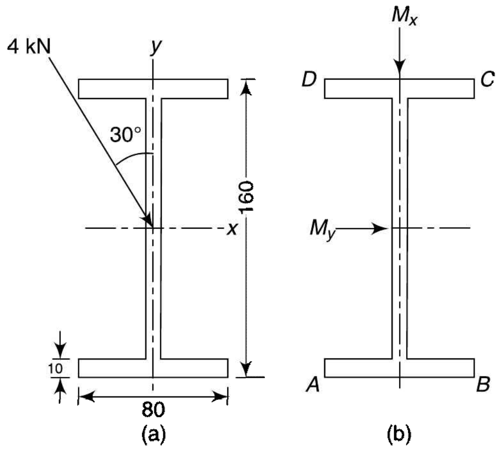


Fig. 5.49

Moment of inertia

$$I_x = 2\left(\frac{80 \times 10^3}{12} + 80 \times 10 \times 75^2\right) + \frac{8 \times 140^3}{12} = 10.843 \times 10^6 \text{ mm}^4$$

$$I_y = 2 \times \frac{10 \times 80^3}{12} + \frac{140 \times 8^3}{12} = 0.859 \times 10^6 \text{ mm}^4$$

Bending moments

$$\text{Maximum bending moment} = \frac{4 \times 2}{4} = 2 \text{ kN·m}$$

$M_x = 2 \times \cos 30° = 1.732$ kN·m (to induce tensile stress in lower half)

$M_y = 2 \times \sin 30° = 1$ kN·m (to induce tensile stress in right half)

Bending stresses

x and y both are positive in the lower right half (4th) quadrant.

The maximum tensile stress occurs at B,

$$\sigma = \frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x = \frac{1.732 \times 10^6}{10.843 \times 10^6} \times 80 + \frac{1 \times 10^6}{0.859 \times 10^6} \times 40 = 59.34 \text{ MPa}$$

The maximum compressive stress occurs at D and is 59.34 MPa

Example 5.36 Figure 5.50 shows a cantilever beam of *I*-section loaded by two inclined loads. Determine the stresses at all the four corners.

Solution

Given A cantilever beam of *I*-section carrying two inclined loads as shown in Fig. 5.50

To find Stresses at all the four corners

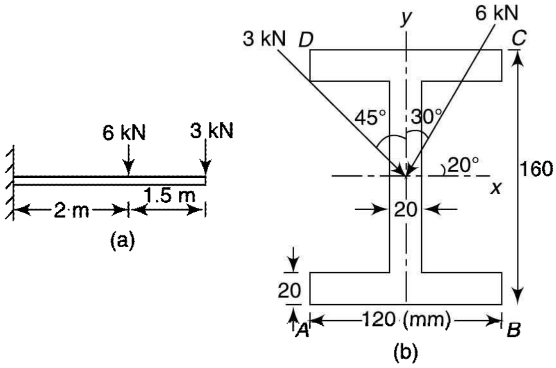


Fig. 5.50

As the section is symmetrical, the centroid is at the centre of the web and the coordinate axes x-x and y-y are also the principal axes (Fig. 5.51a).

Moment of inertia

$$I_x = 2\left(\frac{120 \times 20^3}{12} + 120 \times 20 \times 70^2\right) + \frac{20 \times 120^3}{12} = 26.56 \times 10^6 \text{ mm}^4$$

$$I_y = 2 \times \frac{20 \times 120^3}{12} + \frac{120 \times 20^3}{12} = 5.84 \times 10^6 \text{ mm}^4$$

Bending moments

Remembering that it is a cantilever,

$$M_x = 3 \times 3.5 \cos 45° + 6 \times 2 \cos 30° = 1.732 \text{ kN·m}$$
(to induce tensile stress in upper half)

$$M_y = 3 \times 3.5 \sin 45° - 6 \times 2 \sin 30° = 1.425 \text{ kN·m}$$
(to induce tensile stress in left half)

Thus x and y both are positive in the upper left (2nd) quadrant.

Bending stresses

The maximum tensile stress occurs at D,

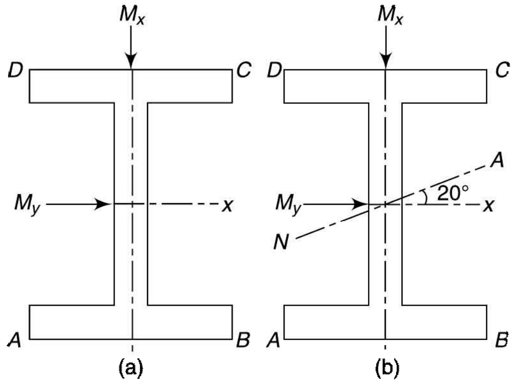


Fig. 5.51

Now, $$\sigma = \frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x$$

$$\sigma_a = -\frac{17.827 \times 10^6}{26.56 \times 10^6} \times 80 + \frac{1.425 \times 10^6}{5.84 \times 10^6} \times 60$$

$$= -0.671 \times 80 + 0.244 \times 60$$

$$= -53.66 + 14.64 = -39.02 \text{ MPa (compressive)}$$

$$\sigma_b = -53.66 - 14.64 = -68.3 \text{ MPa (compressive)}$$

$$\sigma_c = 53.66 - 14.64 = 39.02 \text{ MPa (tensile)}$$

$$\sigma_d = 53.66 + 14.64 = 68.3 \text{ MPa (tensile)}$$

Neutral axis

Inclination of the neutral axis with x-axis is given by,

$$\sigma = \frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x \quad \text{or} \quad -0.671y + 0.244 \cdot x = 0$$

or $$\tan \alpha = \frac{y}{x} = -\frac{0.244}{0.671} = -0.364 \quad \text{or} \quad \alpha = -20°$$

As it is negative, it is taken at an angle of 20° with the x-axis in such a way that it passes through a quadrant in which either x or y is negative. The neutral axis has been shown in the figure. On upper side of the neutral axis, there will be tensile stress and on the lower compressive stress.

Example 5.37 Figure 5.52 shows the cross-section of a 3 m long simply supported beam of T-section carrying a central load inclined at 30° to the y-axis. Determine the maximum load the beam can sustain if the maximum tensile and compressive stresses are not allowed to exceed 40 MPa and 80 MPa respectively. Locate the neutral axis also. The load passes through the centroid of the section.

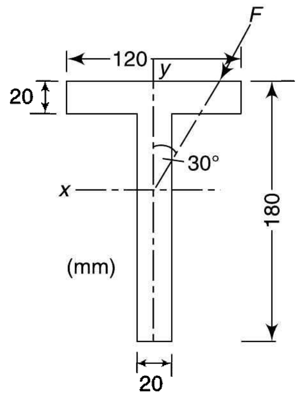


Fig. 5.52

Solution

Given A simply supported beam of T-section as shown in Fig. 5.52. Maximum tensile stress is 40 MPa and maximum compressive stress is 80 MPa.

$$L = 3 \text{ m}$$

To find

— Maximum load

— Location of neutral axis

Moment of inertia

As the section has y-axis as axis of symmetry, it is a principal axis. The other principal axis will be at a perpendicular through the centroid (Fig. 5.53a).

$$y = \frac{120 \times 20 \times 10 + 20 \times 160 \times 100}{120 \times 20 + 20 \times 160} = 61.43 \text{ mm}$$

$$I_x = \frac{120 \times 20^3}{12} + 120 \times 20 \times 51.43^2 + \frac{20 \times 160^3}{12} + 20 \times 160 \times (100 - 61.43)^2$$

$$= (6.428 + 11.587) \times 10^6 \text{ mm}^4 = 18.015 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{20 \times 120^3}{12} + \frac{160 \times 20^3}{12} = 2.987 \times 10^6 \text{ mm}^4$$

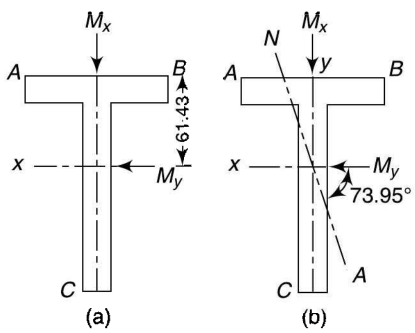


Fig. 5.53

Let F be the maximum load at the centre in kN.

Bending moments

$$\text{Maximum bending moment} = \frac{F \times 3}{4} = 0.75F \text{ kN}\cdot\text{m}$$

$$M_x = 0.75F \cos 30° = 0.65F \text{ kN}\cdot\text{m}$$

(to induce tensile stress in lower half)

$$M_y = 0.75F \sin 30° = 0.375F \text{ kN}\cdot\text{m}$$

(to induce tensile stress in left half)

Thus x and y both are positive in the lower left (3rd) quadrant.

Neutral axis

The equation for the neutral axis is $\sigma = 0$ or $\frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x = 0$

or $$\frac{0.65F \times 10^6}{18.015 \times 10^6} \cdot y + \frac{0.375F \times 10^6}{2.987 \times 10^6} \cdot x = 0$$

or $$0.0361\, F \cdot y + 0.1255\, F \cdot x = 0$$

or $$\tan \alpha = \frac{y}{x} = -\frac{0.1255}{0.0361} = -3.476 \quad \text{or} \quad \theta = -73.95°$$

Neutral axis will pass through 2nd and 4th quadrant as shown in Fig. 5.48b.

Bending stresses

The maximum tensile stress occurs at A or C, i.e.,

$$\sigma_a = 0.0361F \times (-61.43) + 0.1255F \times 60 = -2.216F + 7.533F$$

or $$40 = 5.317\, F \quad \text{or} \quad F = 7.523 \text{ kN}$$

and $$\sigma_c = 0.0361F \times (180 - 61.43) + 0.1255F \times 10 = 4.278F + 1.255F$$

or $$40 = 5.533\, F \quad \text{or} \quad F = 7.229 \text{ kN}$$

Maximum compressive stress will be at B,

$$\sigma_b = -.0361F \times (-61.43) + 0.1255F \times (-60) = -2.216F - 7.533F$$

or $$-80 = -9.749\, F \quad \text{or} \quad F = 8.206 \text{ kN}$$

Thus the maximum load can be 7.229 kN

Example 5.38 Figure 5.54 shows the cross-section of a cantilever channel beam that is 3 m in length. The beam supports a load of 2 kN inclined at 30° to the longitudinal plane at the free end. Locate the position of the neutral axis and find the maximum tensile stress acting on the built-in end. Assume the line of action of load to pass through the shear centre.

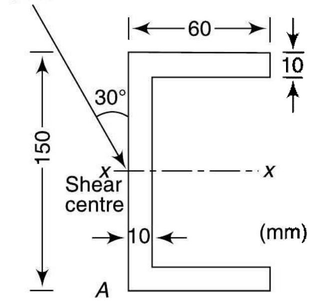


Fig. 5.54

Solution

Given A cantilever channel as shown in Fig. 5.54. As line of action of load passes through shear centre, no twisting moment.

$L = 3$ m $\qquad W = 2$ kN

To find

— Position of neutral axis

— Maximum tensile stress

As the section has x-axis as axis of symmetry, it is a principal axis. The other principal axis will be at right angle through the centroid (Fig. 5.55).

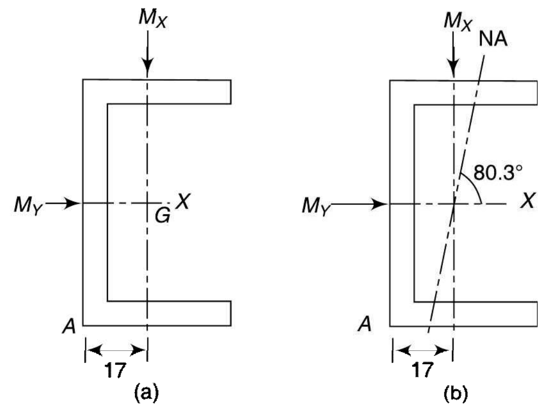


Fig. 5.55

Moment of inertia

$$y = \frac{150 \times 10 \times 5 + 2 \times 50 \times 10 \times 35}{150 \times 10 + 2 \times 50 \times 10} = 17 \text{ mm}$$

$$I_x = \frac{10 \times 150^3}{12} + 2\left(\frac{50 \times 10^3}{12} + 50 \times 10 \times 70^2\right) = 7.721 \times 10^6 \text{ mm}^4$$

I_x can also be found as under:

$$I_x = \frac{60 \times 150^3}{12} - \frac{50 \times 130^3}{12} = 7.721 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{150 \times 10^3}{12} + 150 \times 10 \times (17 - 5)^2 + 2 \times \frac{10 \times 50^3}{12} + 2 \times 10 \times 50 \times (43 - 25)^2$$

$$= 0.761 \times 10^6 \text{ mm}^4$$

or

$$I_y = 2 \times \frac{10 \times 43^3}{3} + \frac{150 \times 17^3}{3} - \frac{130 \times 7^3}{3} = 0.761 \times 10^6 \text{ mm}^4$$

Bending moments

Maximum bending moment = 2 × 3 = 6 kN·m

Resolving into components,

$M_x = 6 \cos 30° = 5.2$ kN·m (due to vertical component of load)

$M_y = 6 \sin 30° = 3$ kN·m (due to horizontal component of load)

As it is a cantilever, both of these components induce tensile stress in the upper left quadrant i.e. above x-axis and left of y-axis in the quadrant containing point A.

Position of neutral axis

The equation for the neutral axis is $\sigma = 0$ or $\frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x = 0$

or $$\frac{5.2 \times 10^6}{7.721 \times 10^6} \cdot y + \frac{3 \times 10^6}{0.761 \times 10^6} \cdot x = 0$$

or $$1.101y + 3.942x = 0$$

or $$\tan \alpha = \frac{y}{x} = -\frac{3.942}{1.101} = -3.58 \quad \text{or} \quad \theta = -74.4°$$

Neutral axis passes through 2nd and 4th quadrant.

Maximum tensile stress

Maximum tensile stress at A =1.101 × 75 + 3.942 × 17 =149.6 MPa

5·7 DETERMINATION OF PRINCIPAL AXES

Sometimes, in case of unsymmetrical sections, the directions of the principal axes are not known. In such cases, the direction of these can be found as follows:

Let OX and OY be any two perpendicular axes through the centroid and OU and OV the principal axes (Fig. 5.56). Also let the inclination of OU with OX be θ.

Let δA be an elemental area and

x and y = coordinate of the area relative to OX, OY

u and v = coordinate of the area relative to OU, OV

Then $u = x \cos \theta + y \sin \theta$

And $v = x \cos \theta + y \sin \theta$

Product of inertia =

$$I_{uv} = \int uv dA = \int (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta)\, dA$$

$$= \int (xy \cos^2 \theta - x^2 \sin \theta \cos \theta + y^2 \sin \theta \cos \theta - xy \sin^2 \theta)\, dA$$

$$= \sin \theta \cos \theta \left(\int y^2 dA - \int x^2 dA\right) + (\cos^2 \theta - \sin^2 \theta) \int xy dA$$

$$= \sin \theta \cos \theta \left(\int y^2 dA - \int x^2 dA\right) + \left(\frac{1 + \cos \theta}{2} - \frac{1 - \cos \theta}{2}\right) \int xy dA$$

$$\left(\frac{1}{2} \sin 2\theta\right)(I_x - I_y) + \cos 2\theta \cdot I_{xy} \qquad (5.19)$$

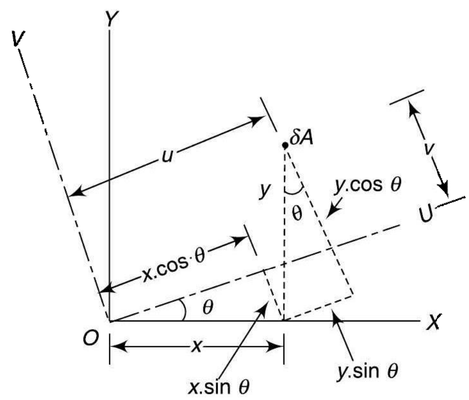


Fig. 5.56

Applying the condition for principal axes, i.e., $I_{uv} = 0$

or $$\left(\frac{1}{2}\sin 2\theta\right)(I_x - I_y) + \cos 2\theta \cdot I_{xy} = 0$$

or $$\sin 2\theta(I_x - I_y) = -2\cos 2\theta \cdot I_{xy} \qquad \text{(i)}$$

or $$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} \qquad (5.20)$$

If I_x, I_y and I_{xy} are calculated, θ can easily be calculated.

The principal moments of inertia I_u and I_v can then be calculated as follows:

Now, $$I_u = \int v^2 \cdot dA$$

$$= \int (y\cos\theta - x\sin\theta)^2 dA = \int (y^2\cos^2\theta + x^2\sin^2\theta - 2xy\sin\theta\cos\theta)dA$$

$$= \cos^2\theta \cdot I_x + \sin^2\theta \cdot I_y - \sin 2\theta \cdot I_{xy} \qquad (5.21)$$

$$= \frac{1+\cos 2\theta}{2}\cdot I_x + \frac{1-\cos 2\theta}{2}\cdot I_y - \sin 2\theta \cdot \frac{\sin 2\theta(I_x - I_y)}{-2\cos 2\theta} \qquad \text{[From (i)]}$$

$$= \frac{1}{2}(I_x + I_y) + \frac{1}{2}\cos 2\theta\,(I_x - I_y) + \frac{\sin^2 2\theta\,(I_x - I_y)}{2\cos 2\theta}$$

$$= \frac{1}{2}(I_x + I_y) + (I_x - I_y)\frac{\cos^2 2\theta + \sin^2 2\theta}{2\cos 2\theta}$$

$$= \frac{1}{2}[(I_x + I_y) + \sec 2\theta\,(I_x - I_y)] \qquad (5.21a)$$

In case I_x and I_y are equal, angle 2θ is 90° and $\cos 2\theta = 0$. In such cases Eq. 5.21 should be used to find I_u as Eq. 5.21a involves division by zero and thus accurate results are not obtained.

$$I_v = \int u^2 \cdot dA$$

$$= \int (x\cos\theta + y\sin\theta)^2\,dA = \int (x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta)\,dA$$

$$= \cos^2\theta \cdot I_y + \sin^2\theta \cdot I_x + \sin 2\theta \cdot I_{xy} \qquad (5.22)$$

Simplifying as above,

$$I_v = \frac{1}{2}[(I_x + I_y) - \sec 2\theta(I_x - I_y)] \qquad (5.22a)$$

Adding 5.21a and 5.22a,

$$I_u + I_v = I_x + I_y \qquad (5.23)$$

I_{xy} for a rectangle with sides parallel to the principal axes can be found as below (Fig. 5.57):

Let h and k be the x- and y-coordinates of the centroid of the rectangle, then

$$I_{xy} = \iint xy \cdot dy \cdot dx = \left(\frac{x^2}{2}\right)_{k-b/2}^{k+b/2} \times \left(\frac{y^2}{2}\right)_{h-d/2}^{h+d/2} = kb \times hd = bd \cdot hk = A \cdot hk \qquad (5.24)$$

If the origin is at the centroid of the rectangle, $I_{xy} = 0$ as h and k are both zero.

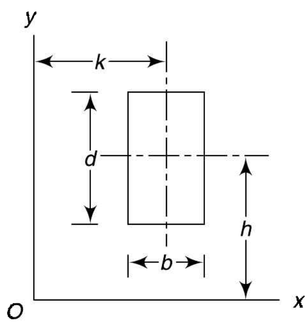


Fig. 5.57

Procedure for the Analysis of Bending Stress in Unsymmetrical Sections

- Locate the x- and y-axis by taking moments of the areas about the vertical and horizontal edges.
- Calculate I_x, I_y and I_{xy}.
- Locate the u-axis by using Eq. 5.20. As this equation is derived by taking u-axis in the counter-clockwise direction with x-axis, a positive value of θ is to be taken counter-clockwise and negative in the clockwise direction to locate u-axis relative to x-axis.
- Find I_u from Eq. 5.21 or 5.21a. If I_x and I_y are equal Eq. 5.21 will give the accurate result. Find I_v using Eqs. 5.22, 5.22a or 5.23.
- Find M and its components M_u and M_v.
- In the four quadrants made by u and v-axes, mark the one in which both components give tensile stresses. This can be judged by considering the load component relative to u and v axes as well as whether the beam is simply supported or is a cantilever. u and v both are taken positive in that quadrant. Similarly, mark the quadrant in which both components give compressive stresses and thus both u and v are negative. In the rest of the two quadrants, mark the sign of u and v according to that on one side of u-axis v is to be positive and on the other negative. Similarly, on one side of v-axis u is to be positive and on the other negative.
- Use the relation $\sigma = \dfrac{M_u}{I_u} \cdot v + \dfrac{M_v}{I_v} \cdot u$ to find the stress at any point in any quadrant by inserting various values.
- Inclination of the neutral axis can be found from the relation, $\tan^{-1}\alpha = v/u$. It gives inclination with the u-axis. A negative value of the angle would mean the axes passes through a quadrant in which either u or v is negative. On one side of the neutral axis, the stresses are to be tensile and on the other compressive.

Example 5.39 An angle beam of 300 mm × 300 mm size is loaded as shown in Fig. 5.58. Determine the direction of the neutral axis and the bending stresses at *A*, *B* and *C*. The line of action of the load passes through the shear centre of the cross-section.

Solution

Given An angle beam as shown in Fig. 5.58.

To find

— Position of neutral axis

— Bending stresses at A, B and C

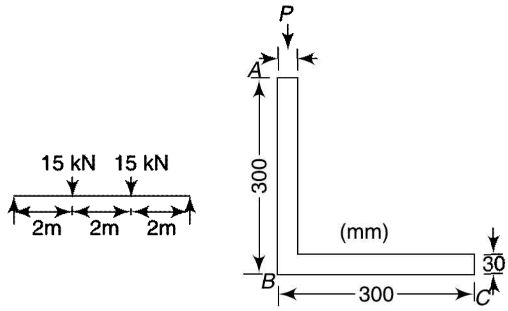


Fig. 5.58

As line of action of load passes through the shear centre, no twisting is there.

Moment of inertia about x- and y-axis

$$y = \frac{300 \times 30 \times 150 + 270 \times 30 \times 15}{300 \times 30 + 270 \times 30} = 86.05 \text{ mm}$$

As it is symmetric angle section, x = 86.05 mm (Fig. 5.59)

$$I_x = \frac{300 \times 30^3}{12} + 300 \times 30 \times 71.05^2 + \frac{30 \times 270^3}{12} + 30 \times 270 \times 78.95^2$$

$$= 145.804 \times 10^6 \text{ mm}^4$$

Due to symmetry, $I_y = 145.804 \times 10^6 \text{ mm}^4$

$$I_{xy} = A \cdot h \cdot k = 300 \times 30(-71.05)(63.95) + 30 \times 270(-71.05)(78.95)$$

$$= -86.329 \times 10^6 \text{ mm}^4$$

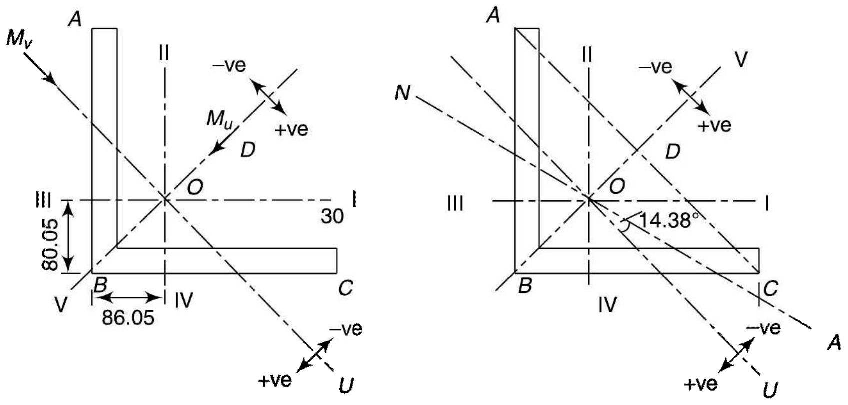


Fig. 5.59

Principal axes

Inclination of principal u-axis,

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} = \frac{2 \times (-86.329) \times 10^6}{0} = -\infty \quad 2\theta = -90^\circ \quad \text{or} \quad \theta = -45^\circ$$

u- and v-axis alongwith four quadrants with respect to these axes are shown in Fig. 5.59.

Moment of inertia about principal axes

I_u can be found from Eq. 5.21,

$$I_u = \cos^2(-45^\circ) \times 145.804 \times 10^6 + \sin^2(-45^\circ) \times 145.804 \times 10^6 - \sin 90^\circ \times (-86.329 \times 10^6)$$

$$= 145.804 \times 10^6 - 86.329 \times 10^6$$

$$= 59.475 \times 10^6 \text{ mm}^4$$

$$I_v = I_x + I_y - I_u = 145.804 \times 10^6 + 145.804 \times 10^6 - 59.475 \times 10^6$$

Bending moments

Maximum bending moment = 15 × 2 = 30 kN·m

M_u = 30 cos 45° = 21.21 kN·m (to induce tensile stress in 3rd and 4th quadrants)

$M_v = M_u$ = 21.21 kN·m (to induce tensile stress in 1st and 4th quadrants)

Thus u and v both are positive in the 4th quadrant.

Bending stresses

For B, $v = OB$ = 2 × 86.05 cos 45° = 121.69 mm; u= 0

$$\sigma_b = \frac{M_u}{I_u} \cdot v + \frac{M_v}{I_v} \cdot u = \frac{21.21 \times 10^6}{59.475 \times 10^6} \times 121.69 + 0 = 43.397 \text{ MPa (tensile)}$$

For A, $v = AD = BD$ = 212.69 mm

$$u = OD = BD - OB = 300 \cos 45^\circ - 121.69 = 212.13 - 121.69 = 90.44 \text{ mm}$$

$$\sigma_a = -\frac{21.21 \times 10^6}{59.475 \times 10^6} \times 90.44 - \frac{21.21 \times 10^6}{232.133 \times 10^6} \times 212.13$$

$$= -0.3566 \times 90.44 - 0.0914 \times 212.13 = -32.253 - 19.382$$

$$= -51.635 \text{ MPa (compressive)}$$

For C, $u = OD = 90.44$ mm; $v = CD = AD = 212.69$ mm; v negative, u positive.

$$\sigma_c = -32.253 + 19.382 = -12.871 \text{ MPa (compressive)}$$

Position of neutral axis

Inclination of the neutral axis, $\tan^{-1}\alpha = \frac{v}{u} = -\frac{0.0914}{0.3566} = -0.2563$

or $\qquad \alpha = -14.38°$

Neutral axis is shown in the Fig. 5.59b. As angle α is negative, the neutral axis has to pass through 1st and 3rd quadrants.

Example 5.40 A 60 mm × 40 mm × 6 mm angle is used as a cantilever with the 40 mm leg horizontal and on the top. The length of the cantilever is 600 mm. Determine the position of the neutral axis and the maximum stress developed if a load of 1 kN is applied at the free end. The line of action of the load passes through the shear centre of the cross-section.

Solution

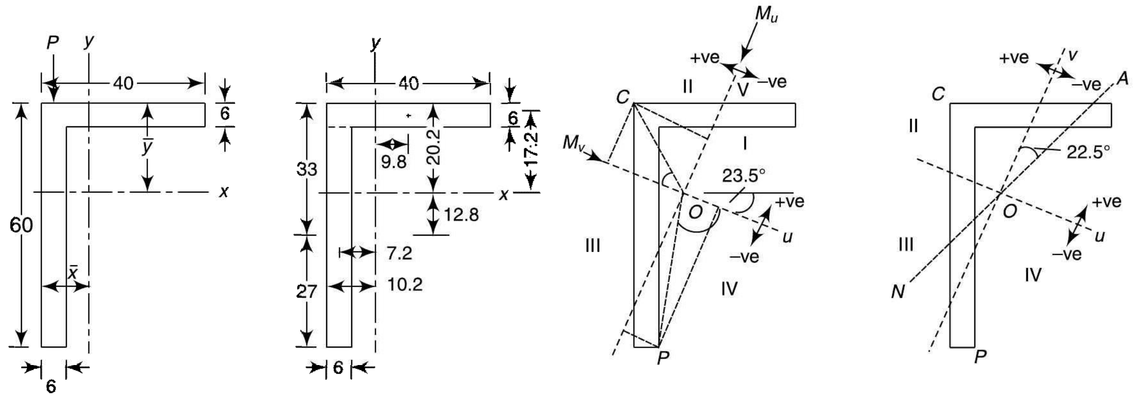


Fig. 5.60

Given An angle beam as shown in Fig. 5.60.

$L = 600$ mm $\qquad W = 1$ kN

To find

— Position of neutral axis

— Maximum stress

Figure 5.60 shows the section and the load.

To locate the centroid, take moments about the left and upper edge,

$$\bar{x} = \frac{40 \times 6 \times 20 + 54 \times 6 \times 3}{40 \times 6 + 54 \times 6} = 10.2 \text{ mm}$$

$$\bar{y} = \frac{40 \times 6 \times 3 + 54 \times 6 \times (27 + 6)}{40 \times 6 + 54\ X\ 6} = 20.2 \text{ mm}$$

Moment of inertia about x-and y-axis

$$I_x = \frac{6 \times 54^3}{12} + 6 \times 54 \times 12.8^2 + \frac{40 \times 6^3}{12} + 40 \times 6 \times 17.2^2 = 203\,537 \text{ mm}^4$$

$$I_y = \frac{54 \times 6^3}{12} + 54 \times 6 \times 7.2^2 + \frac{6 \times 40^3}{12} + 6 \times 40 \times 9.8^2 = 72\,818 \text{ mm}^4$$

$$I_{xy} = 54 \times 6 \times (-7.2)(-12.8) + 40 \times 6 \times 9.8 \times 17.2 = 70\,314 \text{ mm}^4 \qquad \text{...(Eq. 5.24)}$$

Position of principal axes

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} = \frac{2 \times 70\,314}{72\,818 - 203\,537} = -1.0758;\ 2\theta = -47° \text{ or } \theta = -23.5°$$

Take u-axis at 23.5° in the clockwise direction with the x-axis. Four quadrants with respect to u and v axes are also shown in Fig. 5.55.

Moment of inertia about principal axes

$$I_u = \frac{1}{2}[(I_x + I_y) + \sec 2\theta(I_x - I_y)]$$

$$= \frac{1}{2}[(203\,537 + 72\,818) + \sec(2 \times 23.5°)(203\,537 - 72\,818)]$$

$$= \frac{1}{2}(27\,6355 + 191\,671) = 234\,013 \text{ mm}^4$$

$$I_v = \frac{1}{2}(27\,6355 - 191\,671) = 42\,342 \text{ mm}^4$$

Bending moments

Maximum bending moment = 1000 × 600 = 600 × 10^3 N·m

Resolving about uu and vv,

$M_u = 600 \times 10^3 \cos 23.5° = 550.2 \times 10^3$ N·m

(being cantilever, it induces tensile stresses in 1st and 2nd quadrants)

$M_v = 600 \times 10^3 \cos 23.5° = 239.2 \times 10^3$ N·m

(being cantilever, it induces tensile stresses in 2nd and 3rd quadrants)

Thus u and v both are positive in the 2nd quadrant.

Position of neutral axis

$$\sigma = \frac{M_u}{I_u} \cdot v + \frac{M_v}{I_v} \cdot u = \frac{550.2 \times 10^3}{234\,013} \cdot v + \frac{239.2 \times 10^3}{42\,342} \cdot u = 2.351\, v + 5.649\, u$$

The equation for the neutral axis is $\sigma = 0$

or $\quad 2.351\, v + 5.649\, u = 0 \quad$ or $\quad \tan\alpha = \frac{u}{v} = -0.416 \quad$ or $\quad \alpha = -22.5°$

Neutral axis is shown in the Fig. 5.55. As angle α is negative, the neutral axis has to pass through 1st and 3rd quadrants.

Maximum tensile stress

Maximum tensile stress will be at C the coordinates of which can be found as below:

$$OC = \sqrt{10.2^2 + 20.2^2} = 22.63 \text{ mm}$$

$$\angle COD = \sin^{-1}\frac{CD}{OD} = \sin^{-1}\frac{20.2}{22.63} = \sin^{-1} 0.893 = 63.2^o$$

$$\angle COV = 63.2° - 23.5 = 39.7°$$

Thus $v = 22.63 \times \sin 39.7° = 14.46$ mm and $u = 22.63 \times \cos 39.7° = 17.41$ mm
[*Alternatively*, Refer Fig. 5.61,

$$v = CK = EO = DG - GH = 20.2 \cos 23.5° - 10.2 \sin 23.5° = 14.46 \text{ mm}$$

$$u = CE = CD + DE = CD + HO = 20.2 \sin 23.5° + 10.2 \cos 23.5° = 17.41 \text{ mm}]$$

$$\sigma = 2.351\, v + 5.649\, u = 2.351 \times 14.46 + 5.649 \times 17.41 = 132.3 \text{ MPa}$$

Maximum compressive stress

Maximum compressive stress will be at P the coordinates of which are,

$$OC = \sqrt{4.2^2 + 39.8^2} = 40.021 \text{ mm}$$

$$\angle POD = \sin^{-1}\frac{PE}{OP} = \sin^{-1}\frac{39.8}{40.021} = \sin^{-1} 0.9945 = 84°$$

$$\angle POV' = 180° - 84° - 23.5° = 72.5°$$

Thus $v = 40.021 \times \sin 72.5° = 38.17$ mm and $u = 40.021 \times \cos 72.5° = 12.02$ mm
[*Alternatively*, Refer Fig. 5.61,

$$v = PS = PT + TS = PT + QR = 4.2 \sin 23.5° + (60 - 20.2) \cos 23.5° = 38.17 \text{ mm}$$

$$u = PM = SO = RO - RS = 39.8 \sin 23.5° - 4.2 \cos 23.5° = 12.02 \text{ mm}]$$

$$\sigma = 2.351 \times 38.17 + 5.649 \times 12.02 = 147.64 \text{ MPa}$$

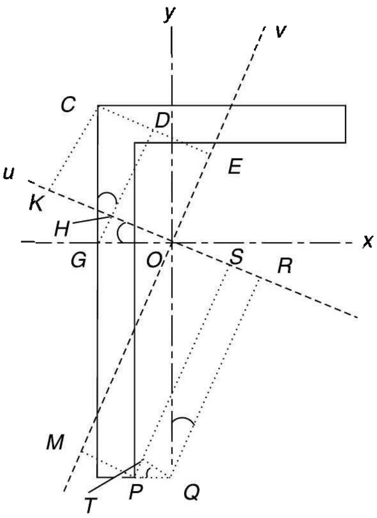


Fig. 5.61

Example 5.41 An angle beam of 140 mm × 100 mm × 10 mm is subjected to a point load of 7 kN as shown in Fig. 5.62. Determine the values of the maximum tensile and compressive bending stresses developed if the line of action of the load passes through the shear centre of the cross-section.

Solution

Given An angle beam as shown in Fig. 5.62.
To find Maximum tensile and compressive stresses

To locate the centroid, take moments about the left and lower edges,

$$\bar{x} = \frac{90 \times 10 \times 55 + 140 \times 10\ X\ 5}{90 \times 10 + 140 \times 10} = 24.56 \text{ mm}$$

$$\bar{y} = \frac{90 \times 10 \times 5 + 140 \times 10 \times 70}{90 \times 10 + 140 \times 10} = 44.56 \text{ mm}$$

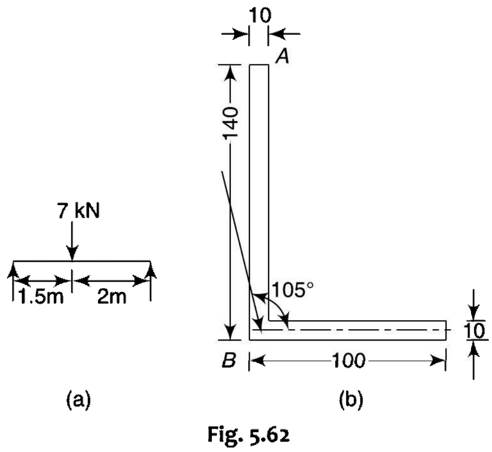


Fig. 5.62

Centroidal axes x and y are shown in Fig. 5.63a.
Moment of inertia about x- and y-axis

$$I_x = \frac{90 \times 10^3}{12} + 90 \times 10 \times 39.56^2 + \frac{10 \times 140^3}{12} + 10 \times 140 \times (70 - 44.56)^2 = 4.609 \times 10^6 \text{ mm}^4$$

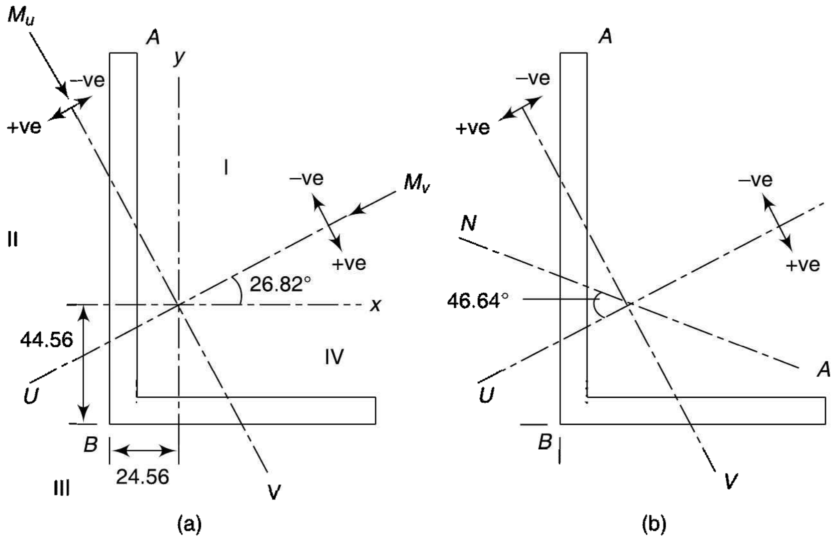


Fig. 5.63

$$I_y = \frac{10 \times 90^3}{12} + 10 \times 90 \times (55 - 24.56)^2 + \frac{140 \times 10^3}{12} + 140 \times 10 \times 19.56^2 = 1.989 \times 10^6 \text{ mm}^4$$

$$I_{xy} = 90 \times 10 \times (55 - 24.56)(-39.56) + 140 \times 10 \times (70 - 44.56)(-19.56) = -1.78 \times 10^6 \text{ mm}^4$$

Position of principal axes

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} = \frac{-2 \times 1.78 \times 10^6}{1.989 \times 10^6 - 4.609 \times 10^6} = 1.359 \quad \text{or} \quad 2\theta = 53.64° \quad \text{or} \quad \theta = 26.82°$$

Take u-axis at 26.82° in the counter-clockwise direction with the x-axis. Four quadrants with respect to u and v axes are also shown in Fig. 5.63a.

Moment of inertia about principal axes

$$I_u = [\cos^2(26.82°) \times 4.609 + \sin^2(26.82°) \times 1.989 - 2 \times (-1.78) \sin 26.82° \times \cos 26.82°]10^6$$

$$= 5.509 \times 10^6 \text{ mm}^4$$

$$I_v = I_x + I_y - I_u = 4.609 \times 10^6 + 1.989 \times 10^6 - 5.509 \times 10^6 = 1.089 \times 10^6 \text{ mm}^4$$

Bending moments

Maximum bending moment on the beam $= R_a \times 1.5 = \frac{7 \times 2}{3.5} \times 1.5 = 6$ kN·m

Angle between the load and u-axis = 105° – 26.56° = 78.18°

Resolving about uu and vv respectively,

$M_u = 6 \sin 78.18° = 5.873$ kN·m (to induce tensile stress in 3rd and 4th quadrants)

$M_v = 6 \cos 78.18° = 1.229$ kN·m (to induce tensile stress in 2nd and 3rd quadrants)

Thus u and v both are positive in the 3rd quadrant.

Position of neutral axis

$$\sigma = \frac{M_u}{I_u}\cdot v + \frac{M_v}{I_v}\cdot u = \frac{5.873\times10^6}{5.509\times10^6}\cdot v + \frac{1.229\times10^3}{1.089}\cdot u = 1.066\,v + 1.129\,u$$

The equation for the neutral axis is $\sigma = 0$

or $$1.066\,v + 1.129\,u = 0 \quad \text{or} \quad \tan^{-1}\alpha = \frac{v}{u} = -\frac{1.129}{1.066} = -1.059 \ \text{or} = -46.64°$$

Neutral axis is shown in the Fig. 5.63. As angle α is negative, the neutral axis has to pass through 2nd and 4th quadrants.

Maximum tensile stress

Maximum tensile stress will be at B the coordinates of which can be found as below:

For B (Fig. 5.64),

$$OB = \sqrt{24.56^2 + 44.56^2} = 50.88 \text{ mm}$$

$$\angle BOD = \sin^{-1}\frac{BD}{OB} = \sin^{-1}\frac{44.56}{50.88} = \sin^{-1}0.8758 = 61.14°$$

$$\angle UOB = 61.14° - 26.82° = 34.32°$$

Thus $v = 50.88 \sin 34.32° = 28.69$ mm and $u = 50.88 \cos 34.32° = 42.02$ mm

$$\sigma = 1.066 \times 28.69 + 1.129 \times 42.02 = 78.02 \text{ MPa}$$

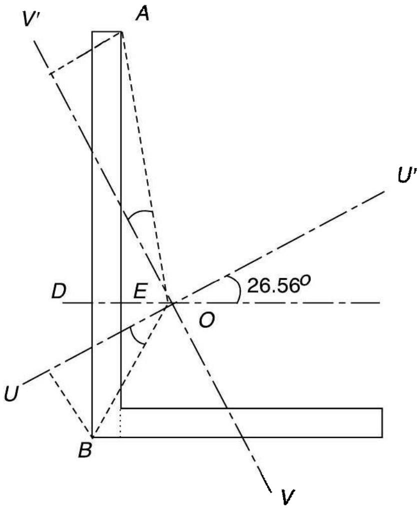


Fig. 5.64

Maximum compressive stress

Maximum compressive stress will be at A the coordinates of which are,

$$OA = \sqrt{14.56^2 + (140 - 44.56)^2} = 96.544 \text{ mm}$$

$$\angle AOE = \cos^{-1}\frac{OE}{OA} = \cos^{-1}\frac{14.56}{96.544} = \cos^{-1}0.1508 = 81.33°$$

$$\angle AOV' = 81.33° - (90° - 26.82°) = 18.15°$$

Thus $v = 96.544 \times \cos 18.15° = 91.74$ mm and $u = 96.544 \times \sin 18.15° = 30.07$ mm

$$\sigma = 1.066 \times 91.74 + 1.129 \times 30.07 = 131.74 \text{ MPa}$$