7 Fourier Series

7.1 INTRODUCTION

We have seen in Chapter 1 that nonhomogeneous differential equations with constant coefficients containing sinusoidal input functions (e.g., $A \sin \omega t$) can be solved quite easily for any input frequency ω. There are many examples, however, of periodic input functions that are not sinusoidal. Figure 7.1 illustrates four common ones. The voltage input to a circuit or the force on a spring–mass system may be periodic but possess discontinuities such as those illustrated. The object of this chapter is to present a technique for solving such problems and others connected to the solution of certain boundary-value problems in the theory of partial differential equations.

The technique of this chapter employs series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right) \tag{7.1.1}$$

the so-called *trigonometric series.* Unlike power series, such series present many pitfalls and subtleties. A complete theory of trigonometric series is beyond the scope of this text and most works on applications of mathematics to the physical sciences. We make our task tractable by narrowing our scope to those principles that bear directly on our interests.

Let $f(t)$ be sectionally continuous in the interval $-T < t < T$ so that in this interval $f(t)$ has at most a finite number of discontinuities. At each point of discontinuity the right- and left-hand limits exist; that is, at the end points $-T$ and T of the interval $-T < t < T$ we define

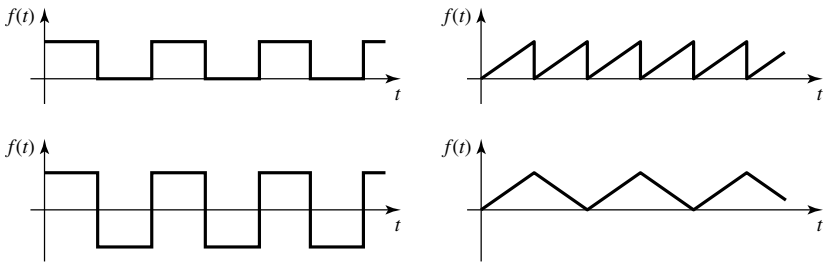


Figure 7.1 Some periodic input functions.

M. C. Potter et al., *Advanced Engineering Mathematics*,
https://doi.org/10.1007/978-3-030-17068-4_7

$f(-T^+)$ and $f(T^-)$ as limits from the right and left, respectively, according to the following expressions:

$$f(-T^+) = \lim_{\substack{t \to -T \\ t > -T}} f(t), \qquad f(T^-) = \lim_{\substack{t \to T \\ t < T}} f(t) \tag{7.1.2}$$

and insist that $f(-T^+)$ and $f(T^-)$ exist also. Then the following sets of *Fourier coefficients* of $f(t)$ in $-T < t < T$ exist:

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{-T}^{T} f(t)\,dt \\ a_n &= \frac{1}{T}\int_{-T}^{T} f(t)\cos\frac{n\pi t}{T}\,dt \\ b_n &= \frac{1}{T}\int_{-T}^{T} f(t)\sin\frac{n\pi t}{T}\,dt, \qquad n = 1, 2, 3, \ldots \end{aligned} \tag{7.1.3}$$

The trigonometric series 7.1.1, defined by using these coefficients, is the *Fourier series* expansion of $f(t)$ in $-T < t < T$. In this case we write

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{T} + b_n \sin\frac{n\pi t}{T}\right) \tag{7.1.4}$$

This representation means only that the coefficients in the series are the Fourier coefficients of $f(t)$ as computed in Eq. 7.1.3. We shall concern ourselves in the next section with the question of when "~" may be replaced with "="; conditions on $f(t)$ which are sufficient to permit this replacement are known as *Fourier theorems.*

We conclude this introduction with an example that illustrates one of the difficulties under which we labor. In the next section we shall show that $f(t) = t$, $-\pi < t < \pi$ has the Fourier series representation

$$t = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nt \tag{7.1.5}$$

where the series converges for all t, $-\pi < t < \pi$. Now $f'(t) = 1$. But if we differentiate the series 7.1.5 term by term, we obtain

$$2\sum_{n=1}^{\infty} (-1)^{n+1} \cos nt \tag{7.1.6}$$

which diverges in $-\pi < t < \pi$ since the nth term, $(-i)^{n+1}\cos nt$, does not tend to zero as n tends to infinity. Moreover, it is not even the Fourier series representation of $f'(t) = 1$. This is in sharp contrast to the "nice" results we are accustomed to in working with power and Frobenius series.

In this chapter we will use *Maple* commands from Appendix C, `assume` from Chapter 3, and `dsolve` from Chapter 1. New commands include: `sum` and `simplify/trig`.

7.1.1 *Maple* Applications

It will be useful to compare a function to its Fourier series representation. Using *Maple,* we can create graphs to help us compare. For example, in order to compare Eq. 7.1.5 with $f(t) = t$, we

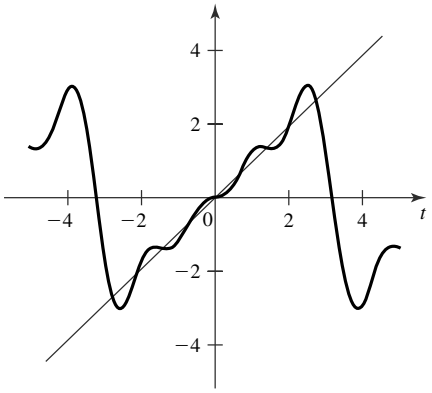
can start by defining a partial sum in *Maple:*

```
>fs:=(N, t) -> sum(2*(-1)^(n+1)* sin(n*t)/n, n=1..N);
```

$$fs := (N,t) \to \sum_{n=1}^{N} \left(\frac{2\,(-1)^{(n+1)}\,\sin(n\ t)}{n} \right)$$

In this way, we can use whatever value of N we want and compare the Nth partial sum with the function $f(t)$:

```
>plot({fs(4, t), t}, t=-5..5);
```



Observe that the Fourier series does a reasonable job of approximating the function only on the interval $-\pi < t < \pi$. We shall see why this is so in the next section.

Problems

1. (a) What is the Fourier representation of $f(t) = 1$, $-\pi < t < \pi$?
(b) Use *Maple* to create a graph of $f(t)$ and a partial Fourier series.

2. Verify the representation, Eq. 7.1.5, by using Eqs. 7.1.3 and 7.1.4.

3. Does the series (Eq. 7.1.5) converge if t is exterior to $-\pi < t < \pi$? At $t = \pi$? At $t = -\pi$? To what values?

4. Show that the Fourier series representation given as Eq. 7.1.4 may be written

$$f(t) \sim \frac{1}{2T}\int_{-T}^{T} f(t)\,dt + \frac{1}{T}\sum_{n=1}^{\infty}\int_{-T}^{T} f(s)\cos\frac{n\pi t}{T}(s-t)\,dt$$

5. Explain how

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

follows from Eq. 7.1.5. *Hint:* Pick $t = \pi/2$. Note that this result also follows from

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots, \qquad -1 < x \le 1$$

6. What is the Fourier series expansion of $f(t) = -1$, $-T < t < T$?

7. Create a graph of $\tan^{-1} x$ and a partial sum, based on the equation in Problem 5.

8. One way to derive Eqs. 7.1.3 is to think in terms of a least squares fit of data (see Section 5.4). In this situation, we let $g(t)$ be the Fourier series expansion of $f(t)$, and we

strive to minimize:

$$\int_{-T}^{T} (f(t) - g(t))^2 \, dt$$

(a) Explain why this integral can be thought of as a function of a_0, a_1, a_2, etc., and b_1, b_2, etc.

(b) Replace $g(t)$ in the above integral with $a_1 \cos(\frac{\pi t}{T})$, creating a function of just a_1. To minimize this function, determine where its derivative is zero, solving for a_1. (Note that it is valid in this situation to switch the integral with the partial derivative.)

(c) Use the approach in part (b) as a model to derive all the equations in Eqs. 7.1.3.

9. ***Computer Laboratory Activity:*** In Section 5.3 in Chapter 5, one problem asks for a proof that for any vectors **y** and **u** (where **u** has norm 1), the projection of the vector **y** in the direction of **u** can be computed by $(\mathbf{u} \cdot \mathbf{y})\mathbf{u}$. We can think of sectionally continuous functions $f(t)$ and $g(t)$, in the interval $-T < t < T$, as vectors, with an inner (dot) product defined by

$$\langle f, g \rangle = \int_{-T}^{T} f(t)g(t) \, dt$$

and a norm defined by

$$||f|| = \sqrt{\langle f, f \rangle}$$

(a) Divide the functions 1, $\cos \dfrac{n\pi t}{T}$, and $\sin \dfrac{n\pi t}{T}$ by appropriate constants so that their norms are 1.

(b) Derive Eqs. 7.1.3 by computing the projections of $f(t)$ in the "directions" of $1, \cos \dfrac{n\pi t}{T}$, and $\sin \dfrac{n\pi t}{T}$.

7.2 A FOURIER THEOREM

As we have remarked in the introduction, we shall assume throughout this chapter that $f(t)$ is sectionally continuous in $-T < t < T$. Whether $f(t)$ is defined at the end points $-T$ or T or defined exterior[1] to $(-T, T)$ is a matter of indifference. For if the Fourier series of $f(t)$ converges to $f(t)$ in $(-T, T)$ it converges almost everywhere since it is periodic with period $2T$. Hence, unless $f(t)$ is also periodic, the series will converge, not to $f(t)$, but to its "periodic extension." Let us make this idea more precise. First, we make the following stipulation:

(1) If t_0 is a point of discontinuity of $f(t)$, $-T < t_0 < T$, then redefine $f(t_0)$, if necessary, so that

$$f(t_0) = \tfrac{1}{2}[f(t_0^-) + f(t_0^+)] \tag{7.2.1}$$

In other words, we shall assume that in $(-T, T)$ the function $f(t)$ is always the average of the right- and left-hand limits at t. Of course, if t is a point of continuity of $f(t)$, then $f(t^+) = f(t^-)$ and hence Eq. 7.2.1 is also true at points of continuity. The *periodic extension* $\tilde{f}(t)$ of $f(t)$ is defined

(2) $$\tilde{f}(t) = f(t), \qquad -T < t < T \tag{7.2.2}$$

(3) $$\tilde{f}(t + 2T) = \tilde{f}(t) \qquad \text{for all } t \tag{7.2.3}$$

(4) $$\tilde{f}(T) = \tilde{f}(-T) = \tfrac{1}{2}[f(-T^+) + f(T^-)] \tag{7.2.4}$$

Condition (2) requires $\tilde{f}(t)$ and $f(t)$ to agree on the *fundamental interval* $(-T, T)$. Condition (3) extends the definition of $f(t)$ so that $\tilde{f}(t)$ is defined everywhere and is periodic with period $2T$. Condition (4) is somewhat more subtle. Essentially, it forces stipulation (1) (see Eq. 7.2.1) on $\tilde{f}(t)$ at the points $\pm nT$ (see Examples 7.2.1 and 7.2.2).

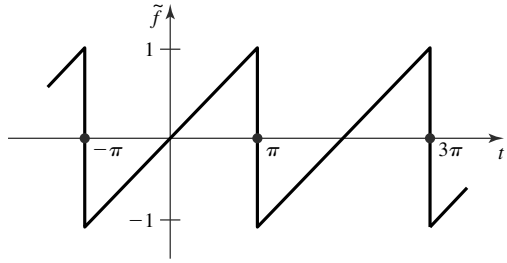
[1]The notation $(-T, T)$ means the set of t, $-T < t < T$. Thus, the exterior of $(-T, T)$ means those t, $t \geq T$ or $t \leq -T$.

EXAMPLE 7.2.1

Sketch the periodic extension of $f(t) = t/\pi, -\pi < t < \pi$.

▶ Solution

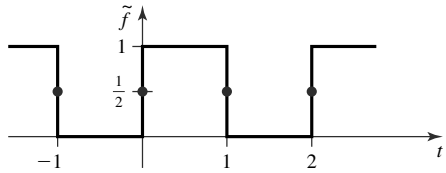
In this example, $f(\pi^-) = 1$ and $f(-\pi^+) = -1$, so that $\tilde{f}(\pi) = \tilde{f}(-\pi) = 0$. The graph of $\tilde{f}(t)$ follows.



Note that the effect of condition (4) (See Eq. 7.2.4) is to force $\tilde{f}(t)$ to have the average of its values at all t; in particular, $\tilde{f}(n\pi) = \tilde{f}(-n\pi) = 0$ for all n.

EXAMPLE 7.2.2

Sketch the periodic extension of $f(t) = 0$ for $t < 0$, $f(t) = 1$ for $t > 0$, if the fundamental interval is $(-1, 1)$.



▶ Solution

There are two preliminary steps. First, we redefine $f(t)$ at $t = 0$; to wit,

$$f(0) = \frac{1+0}{2} = \frac{1}{2}$$

Second, since $f(1) = 1$ and $f(-1) = 0$, we set

$$\tilde{f}(-1) = \tilde{f}(1) = \frac{1+0}{2} = \frac{1}{2}$$

The graph of $f(t)$ is as shown.

A Fourier theorem is a set of conditions sufficient to imply the convergence of the Fourier series $f(t)$ to some function closely "related" to $f(t)$. The following is one such theorem.

Theorem 7.1: *Suppose that $f(t)$ and $f'(t)$ are sectionally continuous in $-T < t < T$. Then the Fourier series of $f(t)$ converges to the periodic extension of $f(t)$, that is, $\tilde{f}(t)$, for all t.*

We offer no proof for this theorem.[2] Note, however, that the Fourier series for the functions given in Examples 7.2.1 and 7.2.2 converge to the functions portrayed in the respective figures of those examples. Thus, Eq. 7.1.4 with an equal sign is a consequence of this theorem.

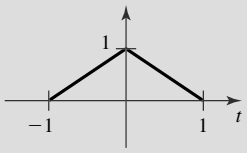
There is another observation relevant to Theorem 7.1; in the interval $-T < t < T$, $\tilde{f}(t) = f(t)$. Thus, the convergence of the Fourier series of $f(t)$ is to $f(t)$ in $(-T, T)$.

[2] A proof is given in many textbooks on Fourier series.

Problems

The following sketches define a function in some interval $-T < t < T$. Complete the sketch for the periodic extension of this function and indicate the value of the function at points of discontinuity.

1.



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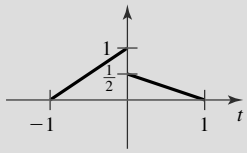
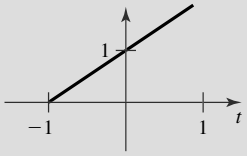


3.



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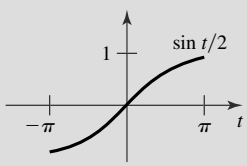

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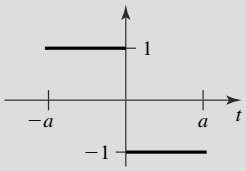
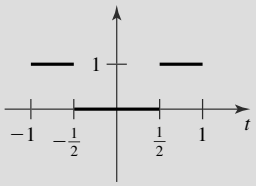
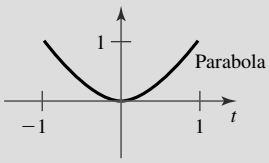


6.



7.



Sketch the periodic extensions of each function.

8. $f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$

9. $f(t) = t + 1, \quad -\pi < t < \pi$

10. $f(t) = \begin{cases} t + \pi, & -\pi < t < 0 \\ -t + \pi, & 0 < t < \pi \end{cases}$

11. $f(t) = |\sin t|, \quad -\pi < t < \pi$

12. $f(t) = \begin{cases} 0, & -2 < t < 0 \\ \sin \pi t/2, & 0 < t < 2 \end{cases}$

13. $f(t) = t^2, \quad -\pi < t < \pi$

14. $f(t) = \begin{cases} -1, & -1 < t < -\frac{1}{2} \\ 0, & -\frac{1}{2} < t < \frac{1}{2} \\ 1, & \frac{1}{2} < t < 1 \end{cases}$

15. $f(t) = |t|, \quad -1 < t < 1$

16. $f(t) = \begin{cases} 0, & -\pi < t < 0 \\ \sin t, & 0 < t < \pi \end{cases}$

17. $f(t) = \begin{cases} -1, & -1 < t < 0 \\ 1, & 0 < t < 1 \end{cases}$

18. $f(t) = \cos t, \quad -\pi < t < \pi$

19. $f(t) = \sin 2t, \quad -\pi < t < \pi$

20. $f(t) = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$

21. $f(t) = t, \ -1 < t < 1$

---

22. Explain why $f(t) = \sqrt{|t|}$ is continuous in $-1 < t < 1$ but $f'(t)$ is not sectionally continuous in this interval.

23. Explain why $f(t) = |t|^{3/2}$ is continuous and $f'(t)$ is also continuous in $-1 < t < 1$. Contrast this with Problem 22.

24. Is $\ln |\tan t/2|$ sectionally continuous in $0 < t < \pi/4$? Explain.

25. Is

$$f(t) = \begin{cases} \ln |\tan t/2|, & 0 < \epsilon \le |t| < \pi/4 \\ 0, & |t| < \epsilon \end{cases}$$

sectionally continuous in $0 < t < \pi/4$? Explain.

7.3 THE COMPUTATION OF THE FOURIER COEFFICIENTS

7.3.1 Kronecker's Method

We shall be faced with integrations of the type

$$\int x^k \cos \frac{n\pi x}{L}\, dx \tag{7.3.1}$$

for various small positive integer values of k. This type of integration is accomplished by repeated integration by parts. We wish to diminish the tedious details inherent in such computations. So consider the integration-by-parts formula

$$\int g(x) f(x)\, dx = g(x) \int f(x)\, dx - \int \left[g'(x) \int f(x)\, dx \right] dx \tag{7.3.2}$$

Let

$$\begin{aligned} F_1(x) &= \int f(x)\, dx \\ F_2(x) &= \int F_1(x)\, dx \\ &\vdots \\ F_n(x) &= \int F_{n-1}(x)\, dx \end{aligned} \tag{7.3.3}$$

Then Eq. 7.3.2 is

$$\int g(x) f(x)\, dx = g(x) F_1(x) - \int g'(x) F_1(x)\, dx \tag{7.3.4}$$

from which

$$\int g(x)f(x)\,dx = g(x)F_1(x) - g'(x)F_2(x) + \int g''(x)F_2(x)\,dx \tag{7.3.5}$$

follows by another integration by parts. This may be repeated indefinitely, leading to

$$\int g(x)f(x)\,dx = g(x)F_1(x) - g'(x)F_2(x) + g''(x)F_3(x) + \cdots \tag{7.3.6}$$

This is Kronecker's method of integration.

Note that each term on the right-hand side of Eq.7.3.6 comes from the preceding term by differentiation of the g function and an indefinite integration of the f function as well as an alternation of sign.

EXAMPLE 7.3.1

Compute $\int_{-\pi}^{\pi} x\cos nx\,dx$.

▶ Solution

We integrate by parts (or use Kronecker's method) as follows:

$$\begin{aligned}\int_{-\pi}^{\pi} x\cos nx\,dx &= \frac{x}{n}\sin nx\Big|_{-\pi}^{\pi} - 1\left(-\frac{1}{n^2}\cos nx\right)\Bigg|_{-\pi}^{\pi}\\ &= 0 + \frac{1}{n^2}(\cos n\pi - \cos n\pi) = 0\end{aligned}$$

EXAMPLE 7.3.2

Compute $\int_{-\pi}^{\pi} x^2\cos nx\,dx$.

▶ Solution

For this example, we can integrate by parts twice (or use Kronecker's method):

$$\begin{aligned}\int_{-\pi}^{\pi} x^2\cos nx\,dx &= \left[\frac{x^2}{n}\sin nx - 2x\left(-\frac{1}{n^2}\cos nx\right) + 2\left(-\frac{1}{n^3}\sin nx\right)\right]_{-\pi}^{\pi}\\ &= \frac{2}{n^2}(\pi\cos n\pi + \pi\cos n\pi) = \frac{4\pi}{n^2}(-1)^n\end{aligned}$$

EXAMPLE 7.3.3

Use Kronecker's method and integrate $\int e^x \cos\ ax\ dx$.

▶ Solution

Let $g(x) = e^x$. Then

$$\begin{aligned}\int e^x \cos ax\, dx &= e^x \frac{1}{a}\sin ax - e^x\left(-\frac{1}{a^2}\cos ax\right) + e^x\left(\frac{-1}{a^3}\sin ax\right) + \cdots \\ &= e^x\left(\frac{1}{a}\sin ax + \frac{1}{a^2}\cos ax - \frac{1}{a^3}\sin ax + \cdots\right) \\ &= e^x \sin ax\left(\frac{1}{a} - \frac{1}{a^3} + \cdots\right) + e^x \cos ax\left(\frac{1}{a^2} - \frac{1}{a^4} + \cdots\right) \\ &= e^x \frac{1}{a}\frac{1}{1+1/a^2}\sin ax + e^x \frac{1}{a^2}\frac{1}{1+1/a^2}\cos ax \\ &= \frac{e^x}{a^2+1}(a\sin ax + \cos ax)\end{aligned}$$

Problems

Find a general formula for each integral as a function of the positive integer n.

1. $\int x^n \cos ax\, dx$
2. $\int x^n \sin ax\, dx$
3. $\int x^n e^{bx}\, dx$
4. $\int x^n \sinh bx\, dx$
5. $\int x^n \cosh bx\, dx$
6. $\int x^n (ax+b)^\alpha\, dx$

Find each integral using as a model the work in Example 7.3.3.

7. $\int e^{bx} \cos ax\, dx$
8. $\int e^{bx} \sin ax\, dx$

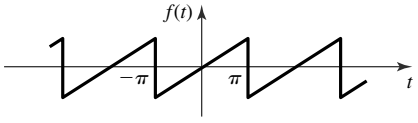
7.3.2 Some Expansions

In this section we will find some Fourier series expansions of several of the more common functions, applying the theory of the previous sections.

EXAMPLE 7.3.4

Write the Fourier series representation of the periodic function $f(t)$ if in one period

$$f(t) = t, \quad -\pi < t < \pi$$



EXAMPLE 7.3.4 *(Continued)*

▶ Solution

For this example, $T = \pi$. For a_n we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt = \frac{t^2}{2\pi}\bigg|_{-n}^{\pi} = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt, \quad n = 1, 2, 3, \ldots$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt = \frac{1}{\pi}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_{-\pi}^{\pi} = 0$$

recognizing that $\cos\, n\pi = \cos(-n\pi)$ and $\sin\, n\pi = -\sin(-n\pi) = 0$. For b_n we have

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt, \quad n = 1, 2, 3, \ldots$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin nt\,dt = \frac{1}{\pi}\left[-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_{-\pi}^{\pi} = -\frac{2}{n}\cos n\pi$$

The Fourier series representation has only sine terms. It is given by

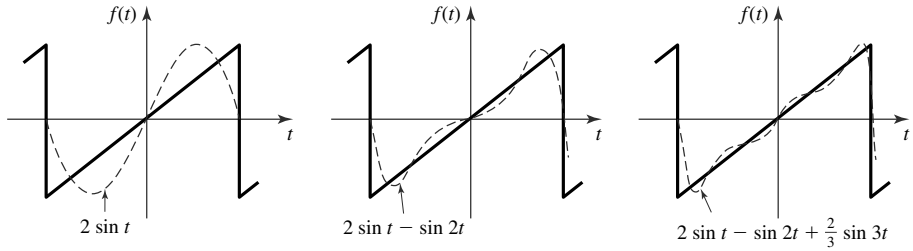
$$f(t) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nt$$

where we have used $\cos\, n\pi = (-1)^n$. Writing out several terms, we have

$$f(t) = -2[-\sin t + \tfrac{1}{2}\sin 2t - \tfrac{1}{3}\sin 3t + \cdots]$$

$$= 2\sin t - \sin 2t + \tfrac{2}{3}\sin 3t - \cdots$$

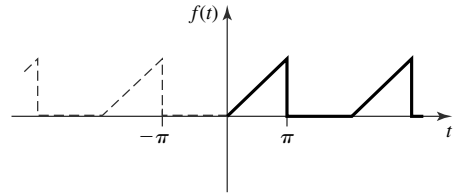
Note the following sketches, showing the increasing accuracy with which the terms approximate the $f(t)$. Notice also the close approximation using three terms. Obviously, using a computer and keeping, say 50 terms, a remarkably good approximation can result using Fourier series.



EXAMPLE 7.3.5

Find the Fourier series expansion for the periodic function $f(t)$ if in one period

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ t, & 0 < t < \pi \end{cases}$$



► Solution

The period is again 2π; thus, $T = \pi$. The Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-0}^{\pi} t\,dt = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0} 0\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} t\cos nt\,dt$$

$$= \frac{1}{\pi}\left[\frac{t}{n}\sin nt + \frac{1}{n^2}\cos nt\right]_0^{\pi} = \frac{1}{\pi n^2}(\cos n\pi - 1), \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0} 0\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} t\sin nt\,dt$$

$$= \frac{1}{\pi}\left[-\frac{t}{n}\cos nt + \frac{1}{n^2}\sin nt\right]_0^{\pi} = -\frac{1}{n}\cos n\pi, \quad n = 1, 2, 3, \ldots$$

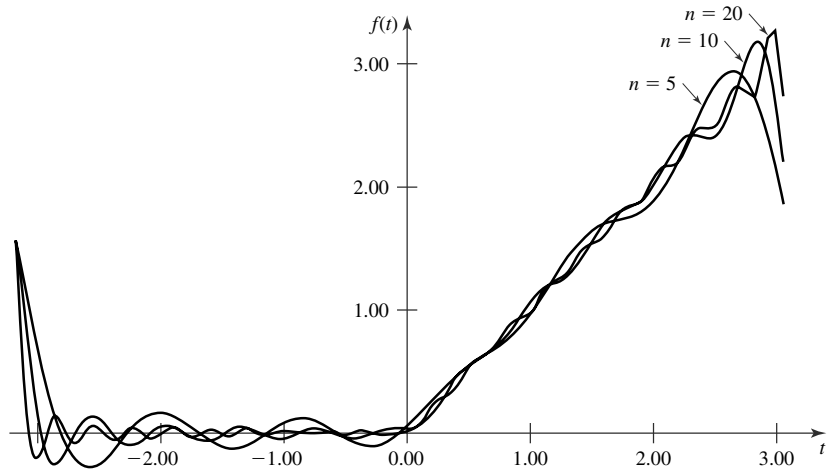
The Fourier series representation is, then, using $\cos n\pi = (-1)^n$,

$$f(t) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nt - \frac{(-1)^n}{n}\sin nt\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\cos t - \frac{2}{9\pi}\cos 3t + \cdots + \sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t + \cdots$$

EXAMPLE 7.3.5 (*Continued*)

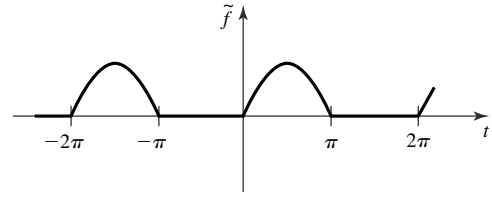
In the following graph, partial fourier series with n equal to 5, 10, and 20, respectively, have been plotted.



EXAMPLE 7.3.6

Find the Fourier series for the periodic extension of

$$f(t) = \begin{cases} \sin t, & 0 \le t \le \pi \\ 0, & \pi \le t \le 2\pi \end{cases}$$



► Solution

The period is 2π and the Fourier coefficients are computed as usual except for the fact that a_1 and b_1 must be computed separately—as we shall see. We have

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin t\, dt = \frac{1}{\pi}(-\cos t)\Big|_0^{\pi} = \frac{2}{\pi}$$

EXAMPLE 7.3.6 *(Continued)*

For $n \neq 1$:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{\pi} \sin t \cos nt\, dt \\
&= \frac{1}{2\pi}\int_0^{\pi} [\sin(t+nt) + \sin(t-nt)]\, dt \\
&= -\frac{1}{2\pi}\left[\frac{\cos(n+1)t}{n+1} - \frac{\cos(n-1)t}{n-1}\right]_0^{\pi} \\
&= -\frac{1}{2\pi}\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right] + \frac{1}{2\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] \\
&= \frac{1}{\pi(n^2-1)}[(-1)^{n+1} - 1] \\
b_n &= \frac{1}{\pi}\int_0^{\pi} \sin t \sin nt\, dt \\
&= \frac{1}{2\pi}\int_0^{\pi} [-\cos(n+1)t + \cos(n-1)t]\, dt \\
&= \frac{1}{2\pi}\left[\frac{-\sin(n+1)t}{n+1} + \frac{\sin(n-1)t}{n-1}\right]_0^{\pi} = 0
\end{aligned}$$

For $n = 1$ the expressions above are not defined; hence, the integration is performed specifically for $n = 1$:

$$\begin{aligned}
a_1 &= \frac{1}{\pi}\int_0^{\pi} \sin t \cos t\, dt \\
&= \frac{1}{\pi}\frac{\sin^2 t}{2}\bigg|_0^{\pi} = 0 \\
b_1 &= \frac{1}{\pi}\int_0^{\pi} \sin t \sin t\, dt = \frac{1}{\pi}\int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2t\right) dt \\
&= \frac{1}{\pi}\left(\frac{1}{2}t - \frac{1}{4}\sin 2t\right)\bigg|_0^{\pi} = \frac{1}{2}
\end{aligned}$$

Therefore, when all this information is incorporated in the Fourier series, we obtain the expansion

$$\begin{aligned}
\tilde{f}(t) &= \frac{1}{\pi} + \frac{1}{2}\sin t + \frac{1}{\pi}\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos nt \\
&= \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nt}{4n^2-1}
\end{aligned}$$

The two series representations for $\tilde{f}(t)$ are equal because $(-1)^{2k+1} - 1 = -2$ and $(-1)^{2k} - 1 = 0$. This series converges everywhere to the periodic function sketched in the example. For $t = \pi/2$, we have

$$\sin\frac{\pi}{2} = \frac{1}{\pi} + \frac{1}{2}\sin\frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}$$

EXAMPLE 7.3.6 (*Continued*)

which leads to

$$\frac{\pi}{4} = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{1}{2} + \frac{1}{3} - \frac{1}{15} + \frac{1}{35} - \frac{1}{63} + \cdots$$

The function $\tilde{f}(t)$ of this example is useful in the theory of diodes.

7.3.3 *Maple* Applications

Clearly the key step in determining Fourier series representation is successful integration to compute the Fourier coefficients. For integrals with closed-form solutions, *Maple* can do these calculations, without n specified, although it helps to specify that n is an integer. For instance, computing the integrals from Example 7.3.6, $n \neq 1$, can be done as follows:

```
>assume(n, integer);
>a_n:=(1/Pi)*(int(sin(t)*cos(n*t), t=0..Pi));
```

$$a\_n := -\frac{(-1)^{n\sim} + 1}{\pi\,(1 + n\sim)(-1 + n\sim)}$$

```
>b_n:=(1/Pi)*(int(sin(t)*sin(n*t), t=0..Pi));
```

$$b\_n := 0$$

Some integrals cannot be computed exactly, and need to be approximated numerically. An example would be to find the Fourier series of the periodic extension of $f(t) = \sqrt{t+5}$ defined on $-\pi \leq t \leq \pi$. A typical Fourier coefficient would be

$$a_3 = \frac{1}{\pi} \int_{-\pi}^{\pi} \sqrt{t+5} \cos 3t \, dt$$

In response to a command to evaluate this integral, *Maple* returns complicated output that involves special functions. In this case, a numerical result is preferred, and can be found via this command:

```
>evalf((1/Pi)*(int(sqrt(t+5)*cos(3*t), t=-Pi..Pi)));
```

```
0.006524598965
```

Problems

Write the Fourier series representation for each periodic function. One period is defined for each. Express the answer as a series using the summation symbol.

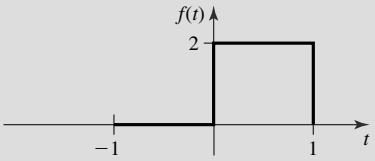
1. $f(t) = \begin{cases} -t, & -\pi < t < 0 \\ t, & 0 < t < \pi \end{cases}$

2. $f(t) = t^2, \quad -\pi < t < \pi$

3. $f(t) = \cos\frac{t}{2}, \quad -\pi < t < \pi$

4. $f(t) = t + 2\pi, \quad -2\pi < t < 2\pi$

5.



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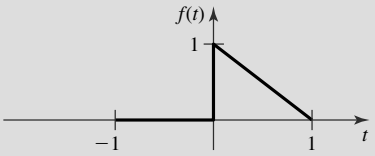

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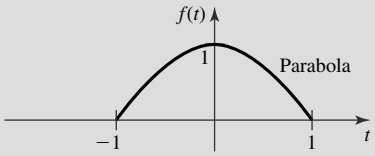

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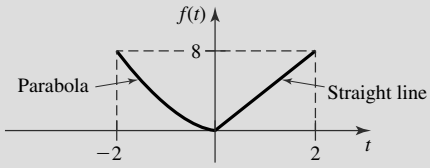


9. Problem 7 of Section 7.2

10. Problem 8 of Section 7.2

11. Problem 9 of Section 7.2

12. Problem 11 of Section 7.2

13. Problem 14 of Section 7.2

Use *Maple* to compute the Fourier coefficients. In addition, create a graph of the function with a partial Fourier series for large N.

14. Problem 1

15. Problem 2

16. Problem 3

17. Problem 4

18. Problem 5

19. Problem 6

20. Problem 7

21. Problem 8

22. Problem 9

23. Problem 10

24. Problem 11

25. Problem 12

26. Problem 13

7.3.4 Even and Odd Functions

The Fourier series expansions of even and odd functions can be accomplished with significantly less effort than needed for functions without either of these symmetries. Recall that an even function is one that satisfies the condition

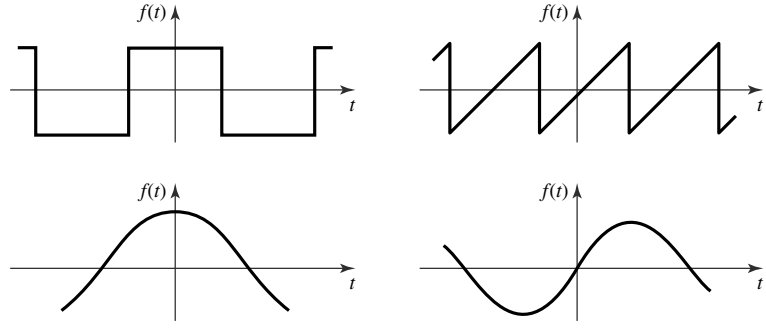
$$f(-t) = f(t) \tag{7.3.7}$$

and hence exhibits a graph symmetric with respect to the vertical axis. An odd function satisfies

$$f(-t) = -f(t) \tag{7.3.8}$$

The functions $\cos t$, $t^2 - 1$, $\tan^2 t$, k, $|t|$ are even; the functions $\sin t$, $\tan t$, t, $t|t|$ are odd. Some even and odd functions are displayed in Fig. 7.2. It should be obvious from the definitions that sums of even (odd) functions are even (odd). The product of two even or two odd functions is even. However, the product of an even and an odd function is odd; for suppose that $f(t)$ is even and $g(t)$ is odd and $h = fg$. Then

$$h(-t) = g(-t)f(-t) = -g(t)f(t) = -h(t) \tag{7.3.9}$$



(a) Even **(b)** Odd

Figure 7.2 Some even and odd functions.

The relationship of Eqs. 7.3.7 and 7.3.8 to the computations of the Fourier coefficients arises from the next formulas. Again, $f(t)$ is even and $g(t)$ is odd. Then

$$\int_{-T}^{T} f(t)\,dt = 2\int_{0}^{T} f(t)\,dt \tag{7.3.10}$$

and

$$\int_{-T}^{T} g(t)\,dt = 0 \tag{7.3.11}$$

To prove Eq. 7.3.10, we have

$$\begin{aligned}\int_{-T}^{T} f(t)\,dt &= \int_{-T}^{0} f(t)\,dt + \int_{0}^{T} f(t)\,dt \\ &= -\int_{T}^{0} f(-s)\,ds + \int_{0}^{T} f(t)\,dt\end{aligned} \tag{7.3.12}$$

by the change of variables $-s = t,\ -ds = dt$. Hence,

$$\begin{aligned}\int_{-T}^{T} f(t)\,dt &= \int_{0}^{T} f(-s)\,ds + \int_{0}^{T} f(t)\,dt \\ &= \int_{0}^{T} f(s)\,ds + \int_{0}^{T} f(t)\,dt\end{aligned} \tag{7.3.13}$$

since $f(t)$ is even. These last two integrals are the same because s and t are dummy variables. Similarly, we prove Eq. 7.3.11 by

$$\begin{aligned}\int_{-T}^{T} g(t)\,dt &= \int_{0}^{T} g(-s)\,ds + \int_{0}^{T} g(t)\,dt \\ &= -\int_{0}^{T} g(s)\,ds + \int_{0}^{T} g(t)\,dt = 0\end{aligned} \tag{7.3.14}$$

because $g(-s) = -g(s)$.

We leave it to the reader to verify:

1. An even function is continuous at $t = 0$, redefining $f(0)$ by Eq. 7.2.1, if necessary.
2. The value (average value, if necessary) at the origin of an odd function is zero.
3. The derivative of an even (odd) function is odd (even).

In view of the above, particularly Eqs. 7.3.10 and 7.3.11, it can be seen that if $f(t)$ is an even function, the *Fourier cosine series* results:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{T} \tag{7.3.15}$$

where

$$a_0 = \frac{2}{T} \int_0^T f(t)\, dt, \qquad a_n = \frac{2}{T} \int_0^T f(t) \cos \frac{n\pi t}{T}\, dt \tag{7.3.16}$$

If $f(t)$ is an odd function, we have the *Fourier sine series,*

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{T} \tag{7.3.17}$$

where

$$b_n = \frac{2}{T} \int_0^T f(t) \sin \frac{n\pi t}{T}\, dt \tag{7.3.18}$$

From the point of view of a physical system, the periodic input function sketched in Fig. 7.3 is neither even or odd. A function may be even or odd depending on where the vertical axis, $t = 0$, is drawn. In Fig. 7.4 we can clearly see the impact of the placement of $t = 0$; it generates an even function $f_1(t)$ in (a), an odd function $f_2(t)$ in (b), and $f_3(t)$ in (c) which is neither even nor odd. The next example illustrates how this observation may be exploited.

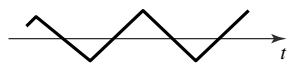
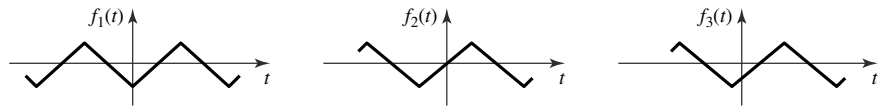


Figure 7.3 A periodic input.

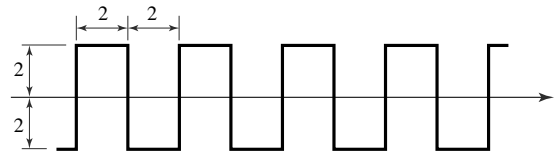


(a) **(b)** **(c)**

Figure 7.4 An input expressed as various functions.

EXAMPLE 7.3.7

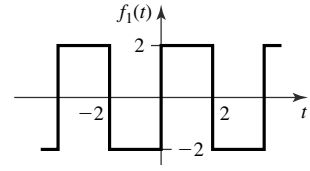
A periodic forcing function acts on a spring–mass system as shown. Find a sine-series representation by considering the function to be odd, and a cosine-series representation by considering the function to be even.



▶ Solution

If the $t = 0$ location is selected as shown, the resulting odd function can be written, for one period, as

$$f_1(t) = \begin{cases} -2 & -2 < t < 0 \\ 2, & 0 < t < 2 \end{cases}$$



For an odd function we know that

$$a_n = 0$$

Hence, we are left with the task of finding b_n. We have, using $T = 2$,

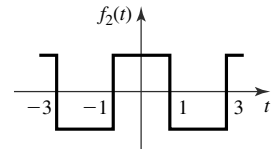
$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T f_l(t)\sin\frac{n\pi t}{T}\,dt,\ n = 1, 2, 3, \ldots \\ &= \frac{2}{2}\int_0^2 2\sin\frac{n\pi t}{2}\,dt = -\frac{4}{n\pi}\cos\frac{n\pi t}{2}\bigg|_0^2 = -\frac{4}{n\pi}(\cos\ n\pi - 1) \end{aligned}$$

The Fourier sine series is, then, again substituting $\cos n\pi = (-1)^n$,

$$\begin{aligned} f_1(t) &= \sum_{n=1}^{\infty}\frac{4[1-(-1)^n]}{n\pi}\sin\frac{n\pi t}{2} \\ &= \frac{8}{\pi}\sin\frac{\pi t}{2} - \frac{8}{3\pi}\sin\frac{3\pi t}{2} + \frac{8}{5\pi}\sin\frac{5\pi t}{2} - \cdots \end{aligned}$$

If we select the $t = 0$ location as displayed, an even function results. Over one period it is

$$f_2(t) = \begin{cases} -2, & -2 < t < -1 \\ 2, & -1 < t < 1 \\ -2, & 1 < t < 2 \end{cases}$$



EXAMPLE 7.3.7 *(Continued)*

For an even function we know that

$$b_n = 0$$

The coefficients a_n are found from

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f_2(t)\cos\frac{n\pi t}{T}\,dt; \qquad n = 1, 2, 3, \ldots \\ &= \frac{2}{2}\left[\int_0^1 2\cos\frac{n\pi t}{2}\,dt + \int_1^2 (-2)\cos\frac{n\pi t}{2}\,dt\right] \\ &= \frac{4}{n\pi}\sin\frac{n\pi t}{2}\bigg|_0^1 - \frac{4}{n\pi}\sin\frac{n\pi t}{2}\bigg|_1^2 = \frac{8}{n\pi}\sin\frac{n\pi}{2} \end{aligned}$$

The result for $n = 0$ is found from

$$\begin{aligned} a_0 &= \frac{2}{T}\int_0^T f_2(t)\,dt \\ &= \frac{2}{2}\left[\int_0^1 2\,dt + \int_1^2 (-2)\,dt\right] = 2 - 2 = 0 \end{aligned}$$

Finally, the Fourier cosine series is

$$\begin{aligned} f_2(t) &= \sum_{n=1}^{\infty}\frac{8}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi t}{2} \\ &= \frac{8}{\pi}\cos\frac{\pi t}{2} - \frac{8}{3\pi}\cos\frac{3\pi t}{2} + \frac{8}{5\pi}\cos\frac{5\pi t}{2} + \cdots \end{aligned}$$

We can take a somewhat different view of the problem in the preceding example. The relationship between $f_1(t)$ and $f_2(t)$ is

$$f_1(t+1) = f_2(t) \tag{7.3.19}$$

Hence, the odd expansion in Example 7.3.7 is just a "shifted" version of the even expansion. Indeed,

$$\begin{aligned} f_1(t+1) = f_2(t) &= \sum_{n=1}^{\infty} 4\frac{[1-(-1)^n]}{n\pi}\sin\frac{n\pi(t+1)}{2} \\ &= \sum_{n=1}^{\infty} 4\frac{[1-(-1)^n]}{n\pi}\left(\sin\frac{n\pi}{2}\cos\frac{n\pi t}{2} + \cos\frac{n\pi}{2}\sin\frac{n\pi t}{2}\right) \\ &= \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos\frac{2n-1}{2}\pi t \end{aligned} \tag{7.3.20}$$

which is an even expansion, equivalent to the earlier one.

Problems

1. In Problems 1 to 8 of Section 7.3.2, (a) which of the functions are even, (b) which of the functions are odd, (c) which of the functions could be made even by shifting the vertical axis, and (d) which of the functions could be made odd by shifting the vertical axis?

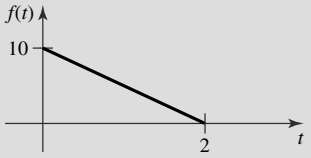
Expand each periodic function in a Fourier sine series and a Fourier cosine series.

2. $f(t) = 4t, \qquad 0 < t < \pi$

3. $f(t) = \begin{cases} 10, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$

4. $f(t) = \sin t, \qquad 0 < t < \pi$

5.



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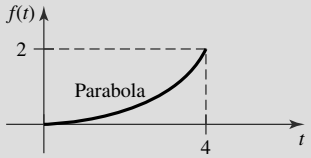


7.

8. Show that the periodic extension of an even function must be continuous at $t = 0$.

9. Show that the period extension of an odd function is zero at $t = 0$.

10. Use the definition of derivative to explain why the derivative of an odd (even) function is even (odd).

Use *Maple* to compute the Fourier coefficients. In addition, create a graph of the function with a partial Fourier series for large N.

11. Problem 2

12. Problem 3

13. Problem 4

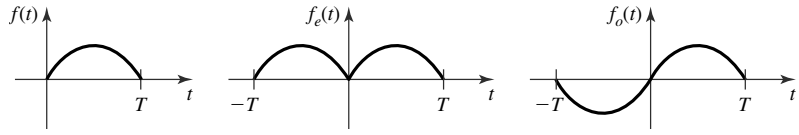
14. Problem 5

15. Problem 6

16. Problem 7

7.3.5 Half-Range Expansions

In modeling some physical phenomena it is necessary that we consider the values of a function only in the interval 0 to T. This is especially true when considering partial differential equations, as we shall do in Chapter 8. There is no condition of periodicity on the function, since there is no interest in the function outside the interval 0 to T. Consequently, we can extend the function arbitrarily to include the interval $-T$ to 0. Consider the function $f(t)$ shown in Fig. 7.5. If we extend it as in part (b), an even function results; an extension as in part (c) results in an odd function. Since these functions are defined differently in $(-T, 0)$ we denote them with different subscripts: f_e for an even extension, f_o for an odd extension. Note that the Fourier series for $f_e(t)$ contains only cosine terms and contains only sine terms for $f_o(t)$. Both series converge to $f(t)$ in $0 < t < T$. Such series expansions are known as *half-range expansions.* An example will illustrate such expansions.



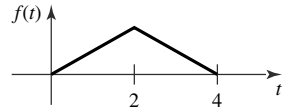
(a) $f(t)$ **(b)** Even function **(c)** Odd function

Figure 7.5 Extension of a function.

EXAMPLE 7.3.8

A function $f(t)$ is defined only over the range $0 < t < 4$ as

$$f(t) = \begin{cases} t, & 0 < t < 2 \\ 4 - t, & 2 < t < 4 \end{cases}$$



Find the half-range cosine and sine expansions of $f(t)$.

▶ Solution

A half-range cosine expansion is found by forming a symmetric extension $f(t)$. The b_n of the Fourier series is zero. The coefficients a_n are

$$\begin{aligned} a_n &= \frac{2}{T}\int_0^T f(t)\cos\frac{n\pi t}{T}\,dt, \qquad n = 1, 2, 3, \cdots \\ &= \frac{2}{4}\int_0^2 t\cos\frac{n\pi t}{4}\,dt + \frac{2}{4}\int_2^4 (4-t)\cos\frac{n\pi t}{4}\,dt \\ &= \frac{1}{2}\left[\frac{4t}{n\pi}\sin\frac{n\pi t}{4} + \frac{16}{\pi^2 n^2}\cos\frac{n\pi t}{4}\right]_0^2 + \frac{1}{2}\left[\frac{16}{n\pi}\sin\frac{n\pi t}{4}\right]_2^4 \\ &\quad - \frac{1}{2}\left[\frac{4t}{n\pi}\sin\frac{n\pi t}{4} + \frac{16}{n^2\pi^2}\cos\frac{n\pi t}{4}\right]_2^4 \\ &= -\frac{8}{n^2\pi^2}\left[1 + \cos n\pi - 2\cos\frac{n\pi}{2}\right] \end{aligned}$$

For $n = 0$ the coefficient a_0 is

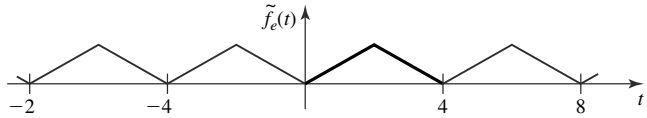
$$a_0 = \tfrac{1}{2}\int_0^2 t\,dt + \tfrac{1}{2}\int_2^4 (4-t)\,dt = 2$$

EXAMPLE 7.3.8 (*Continued*)

The half-range cosine expansion is then

$$f(t) = 1 + \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)\cos\frac{n\pi t}{4}$$
$$= 1 - \frac{8}{\pi^2}\left[\cos\frac{\pi t}{2} + \frac{1}{9}\cos\frac{3\pi t}{2} + \cdots\right], \qquad 0 < t < 4$$

It is an even periodic extension that graphs as follows:



Note that the Fourier series converges for all t, but not to $f(t)$ outside of $0 < t < 4$ since $f(t)$ is not defined there. The convergence is to the periodic extension of the even extension of $f(t)$, namely, $\tilde{f}_e(t)$.

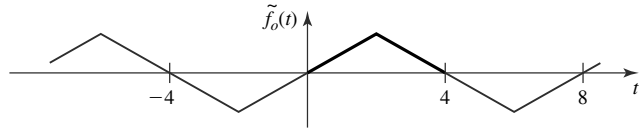
For the half-range sine expansion of $f(t)$, all a_n are zero. The coefficients b_n are

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{n\pi t}{T}\,dt, \qquad n = 1, 2, 3, \ldots$$
$$= \frac{2}{4}\int_0^2 t\,\sin\frac{n\pi t}{4}\,dt + \frac{2}{4}\int_2^4 (4-t)\sin\frac{n\pi t}{4}\,dt = \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

The half-range sine expansion is then

$$f(t) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi t}{4}$$
$$= \frac{8}{\pi^2}\left[\sin\frac{\pi t}{4} - \frac{1}{9}\sin\frac{3\pi t}{4} + \frac{1}{25}\sin\frac{5\pi t}{4} - \cdots\right], \qquad 0 < t < 4$$

This odd periodic extension appears as follows:



Here also we denote the periodic, odd extension of $f(t)$ by $\tilde{f}_o(t)$. The sine series converges to $\tilde{f}_o(t)$ everywhere and to $f(t)$ in $0 < t < 4$. Both series would provide us with good approximations to $f(t)$ in the interval $0 < t < 4$ if a sufficient number of terms are retained in each series. One would expect the accuracy of the sine series to be better than that of the cosine series for a given number of terms, since fewer discontinuities of the derivative exist in the odd extension. This is generally the case; if we make the extension smooth, greater accuracy results for a particular number of terms.

Problems

1. Rework Example 7.3.8 for a more general function. Let the two zero points of $f(t)$ be at $t = 0$ and $t = T$. Let the maximum of $f(t)$ at $t = T/2$ be K.
2. Find a half-range cosine expansion and a half-range sine expansion for the function $f(t) = t - t^2$ for $0 < t < 1$. Which expansion would be the more accurate for an equal number of terms? Write the first three terms in each series.
3. Find half-range sine expansion of

$$f(t) = \begin{cases} t, & 0 < t < 2 \\ 2, & 2 < t < 4 \end{cases}$$

Make a sketch of the first three terms in the series.

Use *Maple* to solve

4. Problem 2
5. Problem 3

7.3.6 Sums and Scale Changes

Let us assume that $f(t)$ and $g(t)$ are periodic functions with period $2T$ and that both functions are suitably[3] defined at points of discontinuity. Suppose that they are sectionally continuous in $-T < t < T$. It can be verified that

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right) \tag{7.3.21}$$

and

$$g(t) \sim \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \left(\alpha_n \cos \frac{n\pi t}{T} + \beta_n \sin \frac{n\pi t}{T} \right) \tag{7.3.22}$$

imply

$$f(t) \pm g(t) \sim \frac{a_0 \pm \alpha_0}{2} + \sum_{n=1}^{\infty} \left[(a_n \pm \alpha_n) \cos \frac{n\pi t}{T} + (b_n \pm \beta_n) \sin \frac{n\pi t}{T} \right] \tag{7.3.23}$$

and

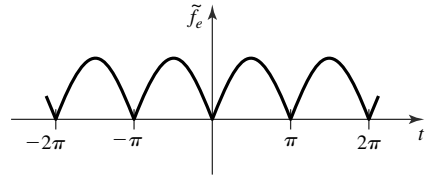
$$cf(t) \sim c\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(ca_n \cos \frac{n\pi t}{T} + cb_n \sin \frac{n\pi t}{T} \right) \tag{7.3.24}$$

These results can often be combined by shifting the vertical axis—as illustrated in Example 7.3.7—to effect an easier expansion.

[3]As before, the value of $f(t)$ at a point of discontinuity is the average of the limits from the left and the right.

EXAMPLE 7.3.9

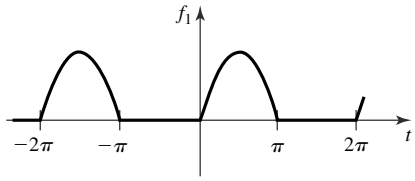
Find the Fourier expansion of the even periodic extension of $f(t) = \sin\ t,\ 0 < t < \pi$, as sketched, using the results of Example 7.3.6.

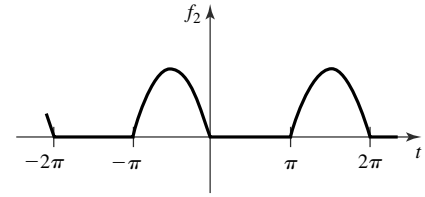


▶ Solution

Clearly, $f_1(t) + f_2(t) = \tilde{f}_e(t)$ as displayed below, where, as usual, $\tilde{f}_e(t)$ represents the even extension of $f(t) = \sin\ t,\ 0 < t < \pi$. But

$$f_1(t + \pi) = f_2(t)$$





and, from Example 7.3.6,

$$f_1(t) = \frac{1}{\pi} + \frac{1}{2}\sin\ t - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos\ 2nt}{4n^2 - 1}$$

Therefore,

$$f_2(t) = f_1(t + \pi) = \frac{1}{\pi} + \frac{1}{2}\sin(t + \pi) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos\ 2n(t + \pi)}{4n^2 - 1}$$

Since $\sin(t + \pi) = -\sin\ t$ and $\cos\ [2n(t + \pi)] = \cos\ 2nt$, we have

$$f_2(t) = \frac{1}{\pi} - \frac{1}{2}\sin\ t - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos\ 2nt}{4n^2 - 1}$$

Finally, without a single integration, there results

$$\begin{aligned}\tilde{f}_e(t) &= f_1(t) + f_2(t)\\ &= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nt}{4n^2 - 1}\end{aligned}$$

It is also useful to derive the effects of a change of scale in t. For instance, if

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi t}{T} + b_n \sin \frac{n\pi t}{T} \right) \tag{7.3.25}$$

then the period of the series is $2T$. Let

$$t = \frac{T}{\tau} \hat{t} \tag{7.3.26}$$

Then

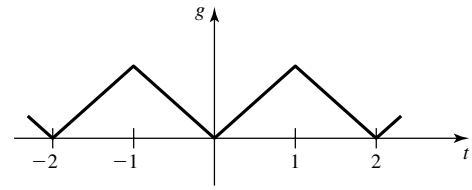
$$\hat{f}(\hat{t}) = f\left(\frac{T}{\tau}\hat{t}\right) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi \hat{t}}{\tau} + b_n \sin \frac{n\pi \hat{t}}{\tau} \right) \tag{7.3.27}$$

is the series representing $\hat{f}(\hat{t})$ with period 2τ. The changes $\tau = 1$ and $\tau = \pi$ are most common and lead to expansions with period 2 and 2π, respectively.

EXAMPLE 7.3.10

Find the Fourier series expansion of the even periodic extension of

$$g(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & 1 \le t < 2 \end{cases}$$



▶ Solution

This periodic input resembles the input in Example 7.3.8. Here the period is 4; in Example 7.3.8 it is 8. This suggests the scale change $2\hat{t} = t$. So if

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 4 - t, & 2 \le t < 4 \end{cases}$$

$$\hat{f}(\hat{t}) = f(2\hat{t}) = \begin{cases} 2\hat{t}, & 0 \le 2\hat{t} < 2 \\ 4 - 2\hat{t}, & 2 \le 2\hat{t} < 4 \end{cases}$$

Note that $g(\hat{t}) = \hat{f}(\hat{t})/2$. So

$$g(\hat{t}) = \begin{cases} \hat{t}, & 0 \le \hat{t} < 1 \\ 2 - \hat{t}, & 1 \le \hat{t} < 2 \end{cases}$$

EXAMPLE 7.3.10 *(Continued)*

But from $g(\hat{t}) = \hat{f}(\hat{t})/2$ we have (see Example 7.3.8)

$$g(\hat{t}) = \frac{1}{2}\left[1 - \frac{8}{\pi^2}\left(\cos \pi\hat{t} + \frac{1}{9}\cos 3\pi\hat{t} + \cdots\right)\right]$$

Replacing $\hat{t}$ by t yields

$$g(t) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi t + \frac{1}{9}\cos\ 3\pi t + \cdots\right), \qquad 0 \le t < 2$$

Problems

1. Let

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ f_1(t), & 0 < t < \pi \end{cases}$$

have the expansion

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

(a) Prove that

$$f(-t) = \begin{cases} f_1(-t), & -\pi < t < 0 \\ 0, & 0 < t < \pi \end{cases}$$

and, by use of formulas for the Fourier coefficients, that

$$f(-t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\ \cos\ nt - b_n \sin nt, \qquad \pi < t < \pi$$

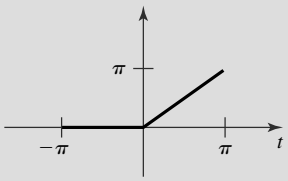
(b) Verify that

$$f_e(t) = a_0 + 2\sum_{n=1}^{\infty} a_n\ \cos\ nt, \qquad -\pi < t < \pi$$

where $f_e(t)$ is the even extension of $f_1(t),\ 0 < t < \pi$.

2. Use the results of Problem 1 and the expansion of

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ t, & 0 < t < \pi \end{cases}$$

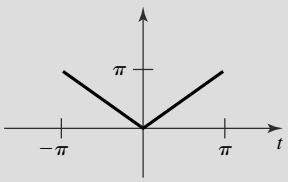


which is

$$\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2}\cos\ nt - \frac{(-1)^n}{n}\sin\ nt$$

to obtain the expansion of

$$f(t) = |t|, \qquad -\pi < t < \pi$$



3. Use the result in Problems 1 and 2 and the methods of this section to find the Fourier expansion of

$$f(t) = \begin{cases} t+1, & -1 < t < 0 \\ -t+1, & 0 < t < 1 \end{cases}$$

4. The Fourier expansion of

$$\hat{f}(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$

is

$$\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(zn-1)t}{2n-1}$$

Use this result to obtain the following expansion:

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases}$$

by observing that $f(t) = [1 + \hat{f}(t)]/2$.

5. Use the information given in Problem 4 and find the expansion of

$$f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 0, & 0 < t < \pi \end{cases}$$

6. If $f(t)$ is constructed as in Problem 1, describe the function $f(t) - f(-t)$.

7. Use Problems 2 and 6 to derive

$$t = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nt, \qquad -\pi < t < \pi$$

7.4 FORCED OSCILLATIONS

We shall now consider an important application involving an external force acting on a spring-mass system. The differential equation describing this motion is

$$M\frac{d^2y}{dt^2} + C\frac{dy}{dt} + Ky = F(t) \tag{7.4.1}$$

If the input function $F(t)$ is a sine or cosine function, the steady-state solution is a harmonic motion having the frequency of the input function. We will now see that if $F(t)$ is periodic with frequency ω but is not a sine or cosine function, then the steady-state solution to Eq. 7.4.1 will contain the input frequency ω and multiples of this frequency contained in the terms of a Fourier series expansion of $F(t)$. If one of these higher frequencies is close to the natural frequency of an underdamped system, then the particular term containing that frequency may play the dominant role in the system response. This is somewhat surprising, since the input frequency may be considerably lower than the natural frequency of the system; yet that input could lead to serious problems if it is not purely sinusoidal. This will be illustrated with an example.

EXAMPLE 7.4.1

Consider the force $F(t)$ acting on the spring–mass system shown. Determine the steady-state response to this forcing function.

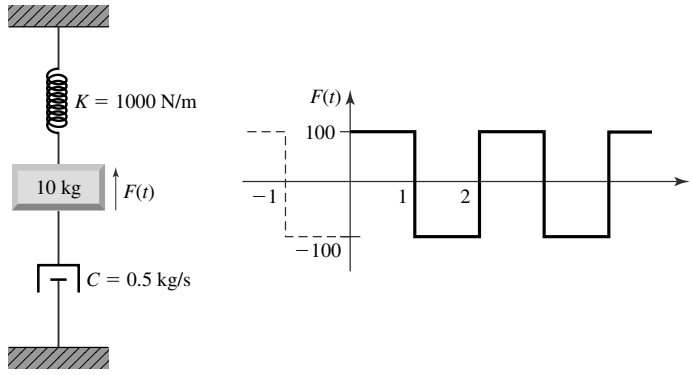
▶ Solution

The coefficients in the Fourier series expansion of an odd forcing function $F(t)$ are (see Example 7.3.7)

$$a_n = 0$$

$$b_n = \frac{2}{1}\int_0^1 100 \sin\frac{n\pi t}{1}\,dt = -\frac{200}{n\pi}\cos n\pi t\Big|_0^1 = -\frac{200}{n\pi}(\cos n\pi - 1), \quad n = 1, 2, \ldots$$

EXAMPLE 7.4.1 (*Continued*)



The Fourier series representation of $F(t)$ is then

$$F(t) = \sum_{n=1}^{\infty} \frac{200}{n\pi}(1 - \cos n\pi) \sin n\pi t = \frac{400}{\pi} \sin \pi t - \frac{400}{3\pi} \sin 3\pi t + \frac{80}{\pi} \sin 5\pi t - \cdots$$

The differential equation can then be written

$$10 \frac{d^2y}{dt^2} + 0.5 \frac{dy}{dt} + 1000y = \frac{400}{\pi} \sin \pi t - \frac{400}{3\pi} \sin 3\pi t + \frac{80}{\pi} \sin 5\pi t - \cdots$$

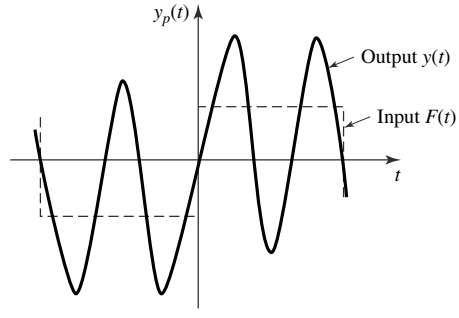
Because the differential equation is linear, we can first find the particular solution $(y_p)_1$ corresponding to the first term on the right, then $(y_p)_2$ corresponding to the second term, and so on. Finally, the steady-state solution is

$$y_p(t) = (y_p)_1 + (y_p)_2 + \cdots$$

Doing this for the three terms shown, using the methods developed earlier, we have

$$(y_p)_1 = 0.141 \sin \pi t - 2.5 \times 10^{-4} \cos \pi t$$
$$(y_p)_2 = -0.376 \sin 3\pi t + 1.56 \times 10^{-3} \cos 3\pi t$$
$$(y_p)_3 = -0.0174 \sin 5\pi t - 9.35 \times 10^{-5} \cos 5\pi t$$

Actually, rather than solving the problem each time for each term, we could have found a $(y_p)_n$ corresponding to the term $[-(200/n\pi)(\cos n\pi - 1) \sin n\pi t]$ as a general function of n. Note the amplitude of the sine term in $(y_p)_2$. It obviously dominates the solution, as displayed in a sketch of $y_p(t)$:



EXAMPLE 7.4.1 (*Continued*)

Yet $(y_p)_2$ has an annular frequency of 3π rad/s, whereas the frequency of the input function was π rad/s. This happened because the natural frequency of the undamped system was 10 rad/s, very close to the frequency of the second sine term in the Fourier series expansion. Hence, it is this overtone that resonates with the system, and not the fundamental. Overtones may dominate the steady-state response for any underdamped system that is forced with a periodic function having a frequency smaller than the natural frequency of the system.

7.4.1 *Maple* Applications

There are parts of Example 7.4.1 that can be solved using *Maple,* while other steps are better done in one's head. For instance, by observing that $F(t)$ is odd, we immediately conclude that $a_n = 0$. To compute the other coefficients:

```
>b[n]:=2*int(100*sin(n*Pi*t), t=0 . .1);
```

$$b_n := -\frac{200\,(\cos(n\pi) - 1)}{n\pi}$$

This leads to the differential equation where the forcing term is an infinite sum of sines. We can now use *Maple* to find a solution for any n. Using `dsolve` will lead to the general solution:

```
>deq:=10*diff(y(t), t$2)+0.5*diff(y(t),
t)+1000*y(t)=b[n]*sin(n*Pi*t);
```

$$deq := 10\left(\frac{d^2}{dt^2}\mathrm{y}(t)\right) + 0.5\left(\frac{d}{dt}\mathrm{y}(t)\right) + 1000\mathrm{y}(t) = -\frac{200\,(\cos(n\pi) - 1)\sin(n\pi t)}{n\pi}$$

```
>dsolve(deq, y(t));
```

$$\begin{aligned}\mathrm{y}(t) = {} & \mathrm{e}^{\left(-\frac{t}{40}\right)}\sin\left(\frac{\sqrt{159999}\,t}{40}\right)\_C2 + \mathrm{e}^{\left(-\frac{t}{40}\right)}\cos\left(\frac{\sqrt{159999}\,t}{40}\right)\_C1 \\ & + (-400000 + 4000n^2\pi^2)\sin(n\pi t - n\pi) + 200n\pi\cos(n\pi t + n\pi) \\ & - 400000\sin(n\pi t + n\pi) + 4000n^2\pi^2\sin(n\pi t + n\pi) \\ & + 800000\sin(n\pi t) - 400\cos(n\pi t)n\pi + 200n\pi\cos(n\pi t - n\pi) \\ & - 8000n^2\pi^2\sin(n\pi t))/(4000000n\pi - 79999n^3\pi^3 + 400n^5\pi^5)\end{aligned}$$

The first two terms of this solution are the solution to the homogeneous equation, and this part will decay quickly as t grows. So, as t increases, any solution is dominated by the particular solution. To get the particular solution, set both constants equal to zero, which can be done with this command:

```
>ypn:= op(3, op(2, %));
```

$$\begin{aligned}ypn := {} & ((-400000 + 4000n^2\pi^2)\sin(n\pi t - n\pi) + 200n\pi\cos(n\pi t + n\pi) \\ & - 400000\sin(n\pi t + n\pi) + 4000n^2\pi^2\sin(n\pi t + n\pi) \\ & + 800000\sin(n\pi t) - 400\cos(n\pi t)n\pi + 200n\pi\cos(n\pi t - n\pi) \\ & - 8000n^2\pi^2\sin(n\pi t))/(4000000n\pi - 79999n^3\pi^3 + 400n^5\pi^5)\end{aligned}$$

This solution is a combination of sines and cosines, with the denominator being the constant:

$$4000000n\pi - 79999n^3\pi^3 + 400n^5\pi^5$$

The following pair of commands can be used to examine the particular solution for fixed values of n. The `simplify` command with the `trig` option combines the sines and cosines. When $n = 1$, we get

```
>subs(n=1, ypn):
>simplify(%, trig);
```

$$-\frac{800\,(-2000\,\sin(\pi t) + 20\pi^2\,\sin(\pi t) + \cos(\pi t)\pi)}{\pi\,(4000000 - 79999\pi^2 + 400\pi^4)}$$

Finally,

```
>evalf(%);
```

$$0.1412659590\,\sin(3.141592654\,t) - 0.0002461989079\,\cos(3.141592654\,t)$$

which reveals $(y_p)_1$ using floating-point arithmetic. Similar calculations can be done for other values of n.

Problems

Find the steady-state solution to Eq. 7.4.1 for each of the following.

1. $M = 2, \quad C = 0, \quad K = 8, \quad F(t) = \sin 4t$

2. $M = 2, \quad C = 0, \quad K = 2, \quad F(t) = \cos 2t$

3. $M = 1, \quad C = 0, \quad K = 16, \quad F(t) = \sin t + \cos 2t$

4. $M = 1, \quad C = 0, \quad K = 25, \quad F(t) = \cos 2t + \frac{1}{10}\sin 4t$

5. $M = 4, \quad C = 0, \quad K = 36, \quad F(t) = \sum_{n=1}^{N} a_n \cos nt$

6. $M = 4, \quad C = 4, \quad K = 36, \quad F(t) = \sin 2t$

7. $M = 1, \quad C = 2, \quad K = 4, \quad F(t) = \cos t$

8. $M = 1, \quad C = 12, \quad K = 16, \quad F(t) = \sum_{n=1}^{N} b_n \sin nt$

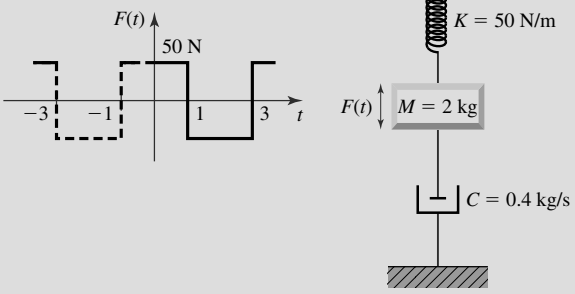
9. $M = 2, \quad C = 2, \quad K = 8, \quad F(t) = \sin t + \frac{1}{10}\cos 2t$

10. $M = 2, \quad C = 16, \quad K = 32,$

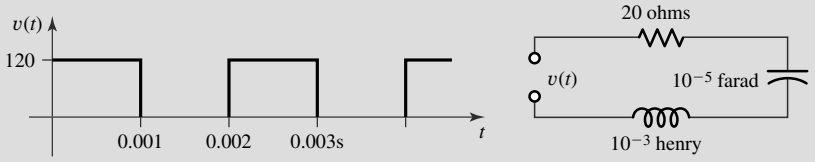
$$F(t) = \begin{cases} t & -\pi/2 < t < \pi/2 \\ \pi - t & \pi/2 < t < 3\pi/2 \end{cases}$$

and $F(t + 2\pi) = F(t)$

11. What is the steady-state response of the mass to the forcing function shown?



12. Determine the steady-state current in the circuit shown.



13. Prove that $(y_p)_n$ from Example 7.4.1 approaches 0 as $n \to \infty$.

Use *Maple* to solve

14. Problem 1

15. Problem 2

16. Problem 3

17. Problem 4

18. Problem 5

19. Problem 6

20. Problem 7

21. Problem 8

22. Problem 9

23. Problem 10

24. Problem 11

25. Problem 12

26. Solve the differential equation in Example 3.8.4 using the method described in this section. Use *Maple* to sketch your solution, and compare your result to the solution given in Example 3.8.4.

27. Solve Problem 12 with Laplace transforms. Use *Maple* to sketch your solution, and compare your result to the solution found in Problem 12.

7.5 MISCELLANEOUS EXPANSION TECHNIQUES

7.5.1 Integration

Term-by-term integration of a Fourier series is a valuable method for generating new expansions. This technique is valid under surprisingly weak conditions, due in part to the "smoothing" effect of integration.

Theorem 7.2: *Suppose that f(t) is sectionally continuous in* $-\pi < t < \pi$ *and is periodic with period* 2π. *Let f(t) have the expansion*

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \tag{7.5.1}$$

Then

$$\int_0^t f(s)\,ds = \sum_{n=1}^{\infty} \frac{b_n}{n} + \sum_{n=1}^{\infty} \left(-\frac{b_n}{n} \cos nt + \frac{a_n}{n} \sin nt \right) \tag{7.5.2}$$

Proof: Set

$$F(t) = \int_0^t f(s)\,ds \tag{7.5.3}$$

and verify $F(t+2\pi) = F(t)$ as follows:

$$\begin{aligned} F(t+2\pi) &= \int_0^{t+2\pi} f(s)\,ds \\ &= \int_0^t f(s)\,ds + \int_t^{t+2\pi} f(s)\,ds \end{aligned} \tag{7.5.4}$$

But $f(t)$ is periodic with period 2π, so that

$$\int_t^{t+2\pi} f(s)\,ds = \int_{-\pi}^{\pi} f(s)\,ds = 0 \tag{7.5.5}$$

since $1/\pi \int_{-\pi}^{\pi} f(s)\,ds = a_0$, which is zero from Eq. 7.5.1. Therefore, Eq. 7.5.4 becomes $F(t+2\pi) = F(t)$. The integral of a sectionally continuous function is continuous from Eq. 7.5.3 and $F'(t) = f(t)$ from this same equation. Hence, $F'(t)$ is sectionally continuous. By the Fourier theorem (Theorem 7.1) we have

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos nt + B_n \sin nt) \tag{7.5.6}$$

valid for all t. Here

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos nt\ dt, \qquad B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin nt\ dt \tag{7.5.7}$$

The formulas 7.5.7 are amenable to an integration by parts. There results

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos nt\ dt \\ &= \frac{1}{\pi}F(t)\frac{\sin nt}{n}\bigg|_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin nt}{n}\,dt \\ &= -\frac{b_n}{n}, \qquad n = 1,\ 2,\ \ldots \end{aligned} \tag{7.5.8}$$

Similarly,

$$\begin{aligned} B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin nt\ dt \\ &= \frac{1}{\pi}F(t)\left(-\frac{\cos nt}{n}\right)\bigg|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\cos nt}{n}\,dt \\ &= \frac{a_n}{n}, \quad n = 1,\ 2,\ \ldots \end{aligned} \tag{7.5.9}$$

because $F(\pi) = F(-\pi + 2\pi) = F(-\pi)$ and $\cos ns = \cos(-ns)$ so that the integrated term is zero. When these values are substituted in Eq. 7.5.6, we obtain

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(-\frac{b_n}{n}\cos nt + \frac{a_n}{n}\sin nt\right) \tag{7.5.10}$$

Now set $t = 0$ to obtain an expression for A_0:

$$F(0) = \int_0^0 f(t)\,dt = 0 = \frac{A_0}{2} - \sum_{n=1}^{\infty} \frac{b_n}{n} \tag{7.5.11}$$

so that

$$\frac{A_0}{2} = \sum_{n=1}^{\infty} \frac{b_n}{n} \tag{7.5.12}$$

Hence, Eq. 7.5.2 is established.

It is very important to note that Eq. 7.5.2 is just the term-by-term integration of relation 7.5.1; one need not memorize Fourier coefficient formulas in Eq. 7.5.2.

EXAMPLE 7.5.1

Find the Fourier series expansion of the even periodic extension of $f(t) = t^2,\ -\pi < t < \pi$. Assume the expansion

$$t = 2\sum_{n=1}^{\infty} \frac{(1)^{n-1}}{n} \sin nt$$

▶ Solution

We obtain the result by integration:

$$\begin{aligned}\int_0^t s\,ds &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \int_0^t \sin ns\,ds \\ &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}(-\cos ns)\Big|_0^t \\ &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} - 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos nt\end{aligned}$$

Of course, $\int_0^t s\,ds = t^2/2$, so that

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} - 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos nt$$

The sum $2\Sigma_{n=1}^{\infty}[(-1)^{n-1}/n^2]$ may be evaluated by recalling that it is $a_0/2$ for the Fourier expansion of $t^2/2$. That is,

$$\begin{aligned}a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{s^2}{2}\,ds &= \frac{1}{\pi}\frac{s^3}{6}\Big|_{-\pi}^{\pi} \\ &= \frac{1}{6\pi}[\pi^3 - (-\pi)^3] = \frac{\pi^2}{3}\end{aligned}$$

EXAMPLE 7.5.1 (*Continued*)

Hence,

$$\frac{a_0}{2} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{6}$$

so

$$t^2 = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} \cos nt$$

EXAMPLE 7.5.2

Find the Fourier expansion of the odd periodic extension of t^3, $-\pi < t < \pi$.

▶ Solution

From the result of Example 7.5.1 we have

$$\frac{t^2}{2} - \frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{-2(-1)^{n-1}}{n^2} \cos nt$$

This is in the form for which Theorem 7.2 is applicable, so

$$\begin{aligned}\int_0^t \left(\frac{s^2}{2} - \frac{\pi^2}{6}\right) ds &= \frac{t^3}{6} - \frac{\pi^2 t}{6} \\ &= -2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin nt\end{aligned}$$

Therefore,

$$t^3 = \pi^2 t - 12\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin nt$$

which is not yet a pure Fourier series because of the $\pi^2 t$ term. We remedy this defect by using the Fourier expansion of t given in Example 7.5.1. We have

$$\begin{aligned}t^3 &= \pi^2 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin nt - 12\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \sin nt \\ &= \sum_{n=1}^{\infty} \left(\frac{2\pi^2}{n} - \frac{12}{n^3}\right) (-1)^{n-1} \sin nt\end{aligned}$$

In summary, note these facts:

1. $\Sigma_{n=1}^{\infty} b_n/n$ converges and is the value $A_0/2$; that is,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F(s)\,ds = \sum_{n=1}^{\infty}\frac{b_n}{n} \tag{7.5.13}$$

2. The Fourier series representing $f(t)$ need not converge to $f(t)$, yet the Fourier series representing $F(t)$ converges to $F(t)$ for all t.
3. If

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \tag{7.5.14}$$

we apply the integration to the function $f(t) - a_0/2$ because

$$f(t) - \frac{a_0}{2} \sim \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \tag{7.5.15}$$

Problems

Use the techniques of this section to obtain the Fourier expansions of the integrals of the following functions.

1. Section 7.2, Problem 1
2. Section 7.2, Problem 3
3. Section 7.2, Problem 5
4. Section 7.2, Problem 6
5. Section 7.2, Problem 9
6. Section 7.2, Problem 13
7. Section 7.2, Problem 14
8. Example 7.3.5
9. Example 7.3.6
10. Section 7.3.2, Problem 4
11. Section 7.3.2, Problem 7
12. Show that we may derive

$$\frac{\pi^2 x - x^3}{12} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n^3}$$

by integration of

$$\frac{\pi^2 - 3x^2}{12} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^2}$$

7.5.2 Differentiation

Term-by-term differentiation of a Fourier series does not lead to the Fourier series of the differentiated function even when that derivative has a Fourier series unless suitable restrictive hypotheses are placed on the given function and its derivatives. This is in marked contrast to term-by-term integration and is illustrated quite convincingly by Eqs. 7.1.4 and 7.1.5. The following theorem incorporates sufficient conditions to permit term-by-term differentiation.

Theorem 7.3: *Suppose that in* $-\pi < t < \pi$, $f(t)$ *is continuous,* $f'(t)$ *and* $f''(t)$ *are sectionally continuous, and* $f(-\pi) = f(\pi)$. *Then*

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \tag{7.5.16}$$

implies that

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left(\frac{a_0}{2}\right) + \sum_{n=1}^{\infty} \frac{d}{dt}(a_n \cos nt + b_n \sin nt) \\ &= \sum_{n=1}^{\infty} nb_n \cos nt - na_n \sin nt \end{aligned} \tag{7.5.17}$$

Proof: We know that df/dt has a convergent Fourier series by Theorem 7.1, in which theorem we use f' for f and f'' for f'. (This is the reason we require f'' to be sectionally continuous.) We express the Fourier coefficients of $f'(t)$ by α_n and β_n so that

$$f'(t) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty} \alpha_n \cos nt + \beta_n \sin nt \tag{7.5.18}$$

where, among other things,

$$\begin{aligned} \alpha_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(s)\,ds \\ &= \frac{1}{\pi}[f(\pi) - f(-\pi)] = 0 \end{aligned} \tag{7.5.19}$$

by hypothesis. By Theorem 7.2, we may integrate Eq. 7.5.18 term by term to obtain

$$\begin{aligned} \int_0^t f'(s)\,ds &= f(t) - f(0) \\ &= \sum_{n=1}^{\infty} \frac{\beta_n}{n} + \sum_{n=1}^{\infty} -\frac{\beta_n}{n}\cos nt + \frac{\alpha_n}{n}\sin nt \end{aligned} \tag{7.5.20}$$

But Eq. 7.5.16 is the Fourier expansion of $f(t)$ in $-\pi < t < \pi$. Therefore, comparing the coefficients in Eqs. 7.5.16 and 7.5.20, we find

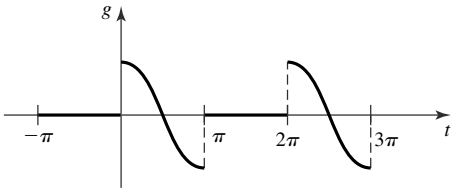
$$a_n = -\frac{\beta_n}{n}, \qquad b_n = \frac{\alpha_n}{n}, \qquad n = 1,\ 2,\ \ldots \tag{7.5.21}$$

We obtain the conclusion (Eq. 7.5.17) by substitution of the coefficient relations (Eq. 7.5.21) into Eq. 7.5.18.

EXAMPLE 7.5.3

Find the Fourier series of the periodic extension of

$$g(t) = \begin{cases} 0, & -\pi < t < 0 \\ \cos t, & 0 < t < \pi \end{cases}$$



▶ Solution

The structure of $g(t)$ suggests examining the function

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ \sin t, & 0 < t < \pi \end{cases}$$

In Example 7.3.6 we have shown that

$$\tilde{f}(t) = \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nt}{4n^2 - 1}$$

Moreover, $f(\pi) = f(-\pi) = 0$ and $f(t)$ is continuous. Also, all the derivatives of $f(t)$ are sectionally continuous. Hence, we may apply Theorem 7.3 to obtain

$$\tilde{g}(t) = \frac{1}{2}\cos t + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{n \sin 2nt}{4n^2 - 1}$$

where $\tilde{g}(t)$ is the periodic extension of $g(t)$. Note, incidentally, that

$$\tilde{g}(0) = \frac{g(0^+) + g(0^-)}{2} = \frac{1}{2}$$

and this is precisely the value of the Fourier series at $t = 0$.

Problems

1. Let $g(t)$ be the function defined in Example 7.5.3. Find $g'(t)$. To what extent does $g'(t)$ resemble

$$f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & -\pi \le t < 0 \end{cases}$$

Differentiate the Fourier series expansion for $g(t)$ and explain why it does not resemble the Fourier series for $-f(t)$.

2. Show that in $-\pi < t < \pi,\ t \neq 0$,

$$\frac{d}{dt}|\sin t| = \begin{cases} -\cos t, & -\pi < t < 0 \\ \cos t, & 0 < t < \pi \end{cases}$$

Sketch $d/dt|\sin t|$ and find its Fourier series. Is Theorem 7.3 applicable?

3. What hypotheses are sufficient to guarantee k-fold term-by-term differentiation of

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

7.5.3 Fourier Series from Power Series[4]

Consider the function $\ln(1+z)$. We know that

$$\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots \tag{7.5.22}$$

is valid for all z, $|z| \leq 1$ except $z = -1$. On the unit circle $|z| = 1$ we may write $z = e^{i\theta}$ and hence,

$$\ln(1+e^{i\theta}) = e^{i\theta} - \tfrac{1}{2}e^{2i\theta} + \tfrac{1}{3}e^{3i\theta} - \cdots \tag{7.5.23}$$

except for $z = -1$, which corresponds to $\theta = \pi$. Now

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{7.5.24}$$

so that $e^{in\theta} = \cos n\theta + i\sin n\theta$ and

$$\begin{aligned} 1 + e^{i\theta} &= 1 + \cos\theta + i\sin\theta \\ &= 2\left(\cos^2\frac{\theta}{2} + i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \\ &= 2\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\cos\frac{\theta}{2} = 2e^{i\theta/2}\cos\frac{\theta}{2} \end{aligned} \tag{7.5.25}$$

Now

$$\ln u = \ln|u| + i\arg u \tag{7.5.26}$$

so that

$$\ln(1+e^{i\theta}) = \ln\left|2\cos\frac{\theta}{2}\right| + i\frac{\theta}{2} \tag{7.5.27}$$

which follows by taking logarithms of Eq. 7.5.25. Thus, from Eqs. 7.5.23, 7.5.24, and 7.5.27, we have

$$\begin{aligned} \ln\left|2\cos\frac{\theta}{2}\right| + i\frac{\theta}{2} = \cos\theta - \frac{1}{2}\cos 2\theta + \cdots \\ + i\left(\sin\theta - \frac{1}{2}\sin 2\theta + \cdots\right) \end{aligned} \tag{7.5.28}$$

and therefore, changing θ to t,

$$\ln\left|2\cos\frac{t}{2}\right| = \cos t - \frac{1}{2}\cos 2t + \frac{1}{3}\cos 3t + \cdots \tag{7.5.29}$$

[4]The material in this section requires some knowledge of the theory of the functions of a complex variable, a topic we explore in Chapter 10.

$$\frac{t}{2} = \sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t + \cdots \tag{7.5.30}$$

Both expansions are convergent in $-\pi < t < \pi$ to their respective functions. In this interval $|2\cos t/2| = 2\cos t/2$ but $\ln(2\cos t/2)$ is not sectionally continuous. Recall that our Fourier theorem is a sufficient condition for convergence. Equation 7.5.29 shows that it is certainly not a necessary one.

An interesting variation on Eq. 7.5.29 arises from the substitution $t = x - \pi$. Then

$$\begin{aligned}\ln\left(2\cos\frac{x-\pi}{2}\right) &= \ln\left(2\sin\frac{x}{2}\right)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\cos n(x-\pi)\\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n}\cos nx\end{aligned} \tag{7.5.31}$$

Therefore, replacing x with t,

$$-\ln\left(2\sin\frac{t}{2}\right) = \sum_{n=1}^{\infty}\frac{1}{n}\cos nt \tag{7.5.32}$$

which is valid[5] in $0 < t < 2\pi$. Adding the functions and their representations in Eqs. 7.5.29 and 7.5.32 yields

$$-\ln\tan\frac{t}{2} = 2\sum_{n=1}^{\infty}\frac{1}{2n-1}\cos(2n-1)t \tag{7.5.33}$$

Another example arises from consideration of

$$\begin{aligned}\frac{a}{a-z} &= \frac{1}{1-z/a}\\ &= 1 + \frac{z}{a} + \frac{z^2}{a^2} + \cdots\\ &= 1 + \frac{\cos\theta}{a} + \frac{\cos 2\theta}{a^2} + \cdots + i\left(\frac{\sin\theta}{a} + \frac{\sin 2\theta}{a^2} + \cdots\right)\end{aligned} \tag{7.5.34}$$

But

$$\begin{aligned}\frac{a}{a-e^{i\theta}} &= \frac{a}{a-\cos\theta - i\sin\theta}\\ &= a\frac{(a-\cos\theta)+i\sin\theta}{(a-\cos\theta)^2+\sin^2\theta}\\ &= a\frac{a-\cos\theta+i\sin\theta}{a^2-2a\cos\theta+1}\end{aligned} \tag{7.5.35}$$

[5]Since $-\pi < t < \pi$ becomes $-\pi < x - \pi < \pi$, we have $0 < x < 2\pi$.

Separating real and imaginary parts and using Eq. 7.5.34 results in the two expansions

$$a\frac{a-\cos t}{a^2-2a\cos t+1}=\sum_{n=0}^{\infty}a^{-n}\cos nt \tag{7.5.36}$$

$$\frac{a\sin t}{a^2-2a\cos t+1}=\sum_{n=1}^{\infty}a^{-n}\sin nt \tag{7.5.37}$$

The expansion are valid for all t, assuming that $a>1$.

Problems

1. Explain why $\ln|2\cos t/2|$ and $\ln(\tan t/2)$ in $-\pi<t<\pi$ or in $0<t<\pi$ are not sectionally continuous.

In each problem use ideas of this section to construct $f(t)$ for the given series.

2. $1+\sum_{n=1}^{\infty}\frac{\cos nt}{n!}$

3. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin 2nt}{(2n)!}$

4. $\sum_{n=1}^{\infty}(-1)^{n}\frac{\cos(2n+1)t}{(2n+1)!}$

5. $1+\sum_{n=1}^{\infty}\frac{\cos\ 2nt}{(2n)!}$

6. Use Eq. 7.5.36 to find the Fourier series expansion of

$$f(t)=\frac{1}{a^2-2a\ \cos\ t+1}$$

Hint: Subtract $\frac{1}{2}$ from both sides of Eq. 7.5.36.

Equations 7.5.36 and 7.5.37 are valid for $a>1$. Find $f(t)$ given

7. $\sum_{n=1}^{\infty}b^n\cos nt, \qquad b<1$

8. $\sum_{n=1}^{\infty}b^n\sin nt, \qquad b<1$

What Fourier series expansions arise from considerations of the power series of each function?

9. $\frac{a}{(a-z)^2}, \qquad a<1$

10. $\frac{a^2}{a^2-z^2}, \qquad a<1$

11. e^{-z}

12. $\sin\ z$

13. $\cosh\ z$

14. $\tan^{-1}\ z$

6 Partial Differential Equations

Outline

6.1 INTRODUCTION

The physical systems studied thus far have been described primarily by ordinary differential equations. We are now interested in studying phenomena that require partial derivatives in the describing equations as they are formed in modeling the particular phenomena. Partial differential equations arise where the dependent variable depends on two or more independent variables. The assumption of lumped parameters in a physical problem usually leads to ordinary differential equations, whereas the assumption of a continuously distributed quantity, a field, generally leads to a partial differential equation. A field approach is quite common now in such undergraduate courses as deformable solids, electromagnetics, and fluid mechanics; hence, the study of partial differential equations is often included in undergraduate programs. Many applications (fluid flow, heat transfer, wave motion) involve second-order equations; thus, they will be emphasized.

The order of the highest derivative is again the order of the equation. The questions of linearity and homogeneity are answered as before in ordinary differential equations. Solutions are superposable as long as the equation is linear. In general, the number of solutions of a partial differential equation is very large. The unique solution corresponding to a particular physical problem is obtained by use of additional information

M. C. Potter, *Engineering Analysis*,
https://doi.org/10.1007/978-3-319-91683-5_6

arising from the physical situation. If this information is given on the boundary as *boundary conditions*, a *boundary-value problem* results. If the information is given at one instant as *initial conditions*, an *initial-value problem* results. A *well-posed problem* has just the right number of these conditions specified to give the solution. We shall not delve into the mathematical theory of making a well-posed problem. We shall, instead, rely on our physical understanding to determine problems that are well posed. We caution the reader that:

1. A problem that has too many boundary and/or initial conditions specified is not well posed and is an overspecified problem.
2. A problem that has too few boundary and/or initial conditions does not possess a unique solution.

In general, a partial differential equation with independent variables x and t which is second order on each of the variables requires two bits of information (this could be dependent on time t) at some x location (or x locations) and two bits of information at some time t, usually $t = 0$.

We are presenting a mathematical tool by way of physical motivation. We shall derive the describing equations of some common phenomena to illustrate the modeling process; other phenomena could have been chosen such as those encountered in magnetic fields, elasticity, fluid flows, aerodynamics, diffusion of pollutants, and so on. An analytical solution technique will then be introduced in this chapter. In a later chapter a numerical technique will be reviewed so that solutions may be obtained to problems that cannot be solved analytically.

We shall be particularly concerned with second-order partial differential equations involving two independent variables, because of the many phenomena that they model. The general form is written as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \tag{6.1.1}$$

where the coefficients may depend on x and y but are most often constants. The equations are classed depending on the coefficients A, B, and C. They are said to be

$$\begin{aligned} &1.\ \text{Elliptic} && \text{if } B^2 - 4AC < 0 \\ &2.\ \text{Parabolic} && \text{if } B^2 - 4AC = 0 \\ &3.\ \text{Hyperbolic} && \text{if } B^2 - 4AC > 0 \end{aligned} \tag{6.1.2}$$

We shall derive equations of each class and illustrate the different types of solutions for each. The boundary conditions are specified depending on the class of the partial differential equation. That is, for an elliptic equation the function (or its derivative) will be specified around the entire boundary enclosing a region of interest, whereas for the hyperbolic and parabolic equations the function cannot be specified around an entire boundary. It is also possible to have an elliptic equation in part of a region of interest and a hyperbolic equation in the remaining part. A discontinuity would separate the two parts of the region; a shock wave would be an example of such a discontinuity.

In the following three sections we shall derive the mathematical equations that describe several phenomena of general interest. The remaining sections will be devoted to the solutions of the equations.

6.2 WAVE MOTION

One of the first phenomena to be modeled with a partial differential equation was that of wave motion. Wave motion occurs in a variety of physical situations; these include vibrating strings, vibrating membranes (drum heads), waves traveling through a solid bar, waves traveling through a solid media (earthquakes), acoustic waves, water waves, compression waves (shock waves), electromagnetic radiation, vibrating beams, and oscillating shafts, to mention a few. We shall illustrate wave motion with several examples.

6.2.1 Vibration of a Stretched, Flexible String

The motion of a tightly stretched, flexible string was modeled with a partial differential equation approximately 250 years ago. It still serves as an excellent introductory example for wave motion. We shall derive the equation that describes the motion and then in later sections present methods of solution.

Suppose that we wish to describe the position for all time of the string shown in Fig. 6.1. In fact, we shall seek a describing equation for the deflection u of the string for any position x and for any time t. The initial and boundary conditions will be considered in detail when the solution is presented.

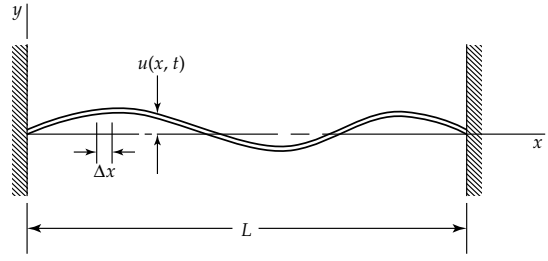


FIGURE 6.1 Deformed, flexible string at an instant t.

Consider an element of the string at a particular instant enlarged in Fig. 6.2. We shall make the following assumptions:

1. The string offers no resistance to bending so that no shearing force exists on a surface normal to the string.

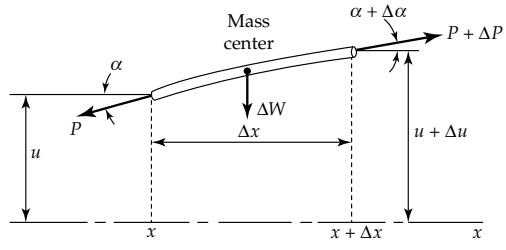


FIGURE 6.2 Small element of the vibrating string.

2. The tension P is so large that the weight of the string is negligible.
3. Every element of the string moves normal to the x axis.

4. The slope of the deflection curve is small.
5. The mass m per unit length of the string is constant.
6. The effects of friction are negligible.

Newton's second law states that the net force acting on a body of constant mass equals the mass M of the body multiplied by the acceleration $\vec{a}$ of the center of mass of the body. This is expressed as

$$\sum \vec{F} = M\vec{a} \tag{6.2.1}$$

Consider the forces acting in the x direction on the element of the string. Using assumption 3 there is no acceleration of the element in the x direction; hence,

$$\sum F_x = 0 \tag{6.2.2}$$

or, referring to Fig. 6.2,

$$(P + \Delta P)\cos(\alpha + \Delta\alpha) - P\cos\alpha = 0 \tag{6.2.3}$$

Using assumption 4 we have

$$\cos\alpha \cong \cos(\alpha + \Delta\alpha) \cong 1 \tag{6.2.4}$$

Equation 6.2.3 then gives us

$$\Delta P = 0 \tag{6.2.5}$$

showing us that the tension is constant along the string.

For the y direction we have, neglecting friction and the weight of the string,

$$P\sin(\alpha + \Delta\alpha) - P\sin\alpha = m\Delta x\frac{\partial^2}{\partial t^2}\left(u + \frac{\Delta u}{2}\right) \tag{6.2.6}$$

where $m\Delta x$ is the mass of the element and $\partial^2/\partial t^2(u + \Delta u/2)$ is the acceleration of the mass center. Again, using assumption 4 we have

$$\begin{aligned} \sin(\alpha + \Delta\alpha) \cong \tan(\alpha + \Delta\alpha) &= \frac{\partial u}{\partial x}(x + \Delta x, t) \\ \sin\alpha \cong \tan\alpha &= \frac{\partial u}{\partial x}(x, t) \end{aligned} \tag{6.2.7}$$

Equation 6.2.6 can then be written as

$$P\left[\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right] = m\Delta x\frac{\partial^2}{\partial t^2}\left(u + \frac{\Delta u}{2}\right) \tag{6.2.8}$$

or, equivalently,

$$P\frac{\dfrac{\partial u}{\partial x}(x + \Delta x, t) - \dfrac{\partial u}{\partial x}(x, t)}{\Delta x} = m\frac{\partial^2}{\partial t^2}\left(u + \frac{\Delta u}{2}\right) \tag{6.2.9}$$

Now, we let $\Delta x \to 0$, which also implies that $\Delta u \to 0$. Then, using the definition,

$$\lim_{\Delta x \to 0} \frac{\dfrac{\partial u}{\partial x}(x + \Delta x, t) - \dfrac{\partial u}{\partial x}(x, t)}{\Delta x} = \frac{\partial^2 u}{\partial x^2}, \tag{6.2.10}$$

our describing equation becomes

$$P\frac{\partial^2 u}{\partial x^2} = m\frac{\partial^2 u}{\partial t^2} \tag{6.2.11}$$

This is usually written in the form

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{6.2.12}$$

where we have set

$$a = \sqrt{\frac{P}{m}} \tag{6.2.13}$$

Equation 6.2.12 is the *one-dimensional wave equation* and a is the *wave speed.* It is a transverse wave; that is, it moves normal to the string. This hyperbolic equation will be solved in a subsequent section.

6.2.2 The Vibrating Membrane

A stretched vibrating membrane, such as a drumhead, is simply an extension into another dimension of the vibrating-string problem. We shall derive a partial differential equation that describes the deflection u of the membrane for any position (x, y) and for any time t. The simplest equation results if the following assumptions are made:

1. The membrane offers no resistance to bending, so shearing stresses are absent.
2. The tension τ per unit length is so large that the weight of the membrane is negligible.
3. Every element of the membrane moves normal to the xy plane.
4. The slope of the deflection surface is small.
5. The mass m of the membrane per unit area is constant.
6. Frictional effects are neglected.

With these assumptions we can now apply Newton's second law to an element of the membrane as shown in Fig. 6.3. Assumption 3 leads to the conclusion that τ is constant throughout the membrane, since there are no accelerations of the element in the x and y directions. This is shown on the element. In the z direction we have

$$\sum F_z = Ma_z \tag{6.2.14}$$

For our element this becomes

$$\tau\,\Delta x \sin(\alpha + \Delta\alpha) - \tau\,\Delta x \sin\alpha + \tau\,\Delta y \sin(\beta + \Delta\beta) - \tau\,\Delta y \sin\beta = m\,\Delta x\,\Delta y \frac{\partial^2 u}{\partial t^2} \tag{6.2.15}$$

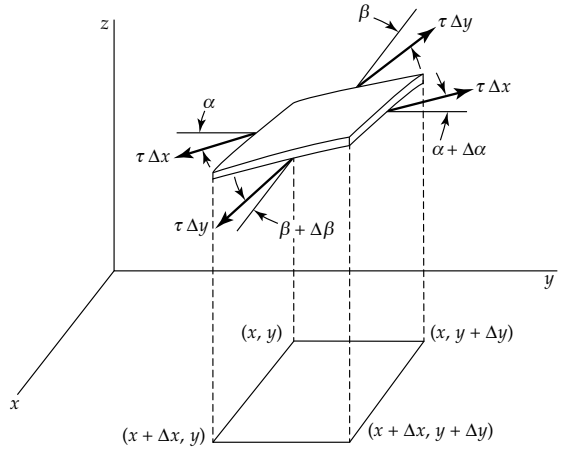


FIGURE 6.3 Element from a stretched, flexible membrane.

where the mass of the element is $m\,\Delta x\,\Delta y$ and the acceleration a_z is $\partial^2 u/\partial t^2$. Recognizing that for small angles

$$\begin{aligned}
\sin(\alpha+\Delta\alpha) \cong \tan(\alpha+\Delta\alpha) &= \frac{\partial u}{\partial y}\left(x+\frac{\Delta x}{2},\ y+\Delta y,\ t\right)\\
\sin\alpha \cong \tan\alpha &= \frac{\partial u}{\partial y}\left(x+\frac{\Delta x}{2},\ y,\ t\right)\\
\sin(\beta+\Delta\beta) \cong \tan(\beta+\Delta\beta) &= \frac{\partial u}{\partial x}\left(x+\Delta x,\ y+\frac{\Delta y}{2},\ t\right)\\
\sin\beta \cong \tan\beta &= \frac{\partial u}{\partial x}\left(x,\ y+\frac{\Delta y}{2},\ t\right)
\end{aligned} \tag{6.2.16}$$

we can then write Eq. 6.2.15 as

$$\begin{aligned}
&\tau\,\Delta x\left[\frac{\partial u}{\partial y}\left(x+\frac{\Delta x}{2}, y+\Delta y, t\right)-\frac{\partial u}{\partial y}\left(x+\frac{\Delta x}{2},\ y, t\right)\right]\\
&\quad+\tau\,\Delta y\left[\frac{\partial u}{\partial x}\left(x+\Delta x,\ y+\frac{\Delta y}{2}, t\right)-\frac{\partial u}{\partial x}\left(x, y+\frac{\Delta y}{2}, t\right)\right]=m\,\Delta x\,\Delta y\,\frac{\partial^2 u}{\partial t^2}
\end{aligned} \tag{6.2.17}$$

or, by dividing by $\Delta x\,\Delta y$,

$$\begin{aligned}
\tau\Bigg[&\frac{\dfrac{\partial u}{\partial y}\left(x+\dfrac{\Delta x}{2},\ y+\Delta y,\ t\right)-\dfrac{\partial u}{\partial y}\left(x+\dfrac{\Delta x}{2},\ y,\ t\right)}{\Delta y}\\
&+\frac{\dfrac{\partial u}{\partial x}\left(x+\Delta x, y+\dfrac{\Delta y}{2},\ t\right)-\dfrac{\partial u}{\partial x}\left(x,\ y+\dfrac{\Delta y}{2},\ t\right)}{\Delta x}\Bigg]=m\frac{\partial^2 u}{\partial t^2}
\end{aligned} \tag{6.2.18}$$

Taking the limit as $\Delta x \to 0$ and $\Delta y \to 0$, we arrive at

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{6.2.19}$$

where

$$a = \sqrt{\frac{\tau}{m}} \tag{6.2.20}$$

Equation 6.2.19 is the *two-dimensional wave equation* and a is the wave speed.

6.2.3 Longitudinal Vibrations of an Elastic Bar

For another example of wave motion, let us determine the equation describing the motion of an elastic bar (steel, for example) that is subjected to an initial displacement or velocity. An example would be striking the bar on the end with a hammer, Fig. 6.4. We make the following assumptions:

1. The bar has a constant cross-sectional area A in the unstrained state.
2. All cross-sectional planes remain plane.
3. The density ρ remains constant throughout the bar.
4. Hooke's law may be used to relate stress and strain.

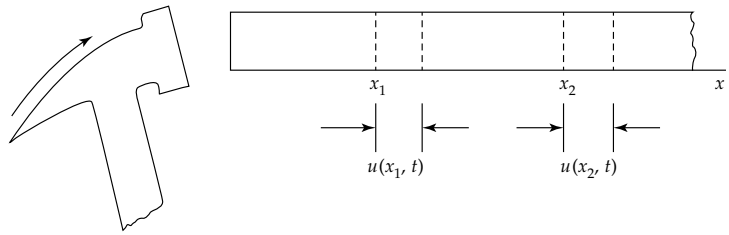


FIGURE 6.4 Wave motion in an elastic bar.

We let $u(x, t)$ denote the displacement of the plane of particles that were at x at $t = 0$. Consider the element of the bar between x_1 and x_2, shown in Fig. 6.5. We assume that the bar has mass ρ per unit volume. The force exerted on the element at x_1 is, by Hooke's law,

$$F_x = \text{area} \times \text{stress} = \text{area} \times E \times \text{strain}, \tag{6.2.21}$$

where E is the modulus of elasticity. The strain ϵ at x_1 is given by

$$\epsilon = \frac{\text{elongation}}{\text{unstrained length}} \tag{6.2.22}$$

Thus, for Δx_1 small, we have the strain at x_1 as

$$\epsilon = \frac{u(x_1 + \Delta x_1, t) - u(x_1, t)}{\Delta x} \tag{6.2.23}$$

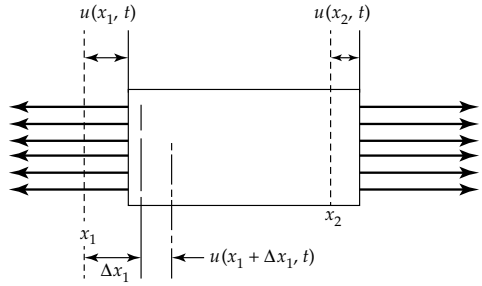


FIGURE 6.5 Element of an elastic bar.

Letting $\Delta x_1 \to 0$, we find that

$$\epsilon = \frac{\partial u}{\partial x} \tag{6.2.24}$$

Returning to the element, the force acting in the x direction is

$$F_x = AE\left[\frac{\partial u}{\partial x}(x_2, t) - \frac{\partial u}{\partial x}(x_1, t)\right] \tag{6.2.25}$$

Newton's second law states that

$$\begin{aligned} F_x &= ma \\ &= \rho A(x_2 - x_1)\frac{\partial^2 u}{\partial t^2} \end{aligned} \tag{6.2.26}$$

Hence, Eqs. 6.2.25 and 6.2.26 give

$$\rho A(x_2 - x_1)\frac{\partial^2 u}{\partial t^2} = AE\left[\frac{\partial u}{\partial x}(x_2, t) - \frac{\partial u}{\partial x}(x_1, t)\right] \tag{6.2.27}$$

We divide Eq. 6.2.27 by $(x_2 - x_1)$ and let $x \to x_2$, to give

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{6.2.28}$$

where the longitudinal wave speed a is given by

$$a = \sqrt{\frac{E}{\rho}} \tag{6.2.29}$$

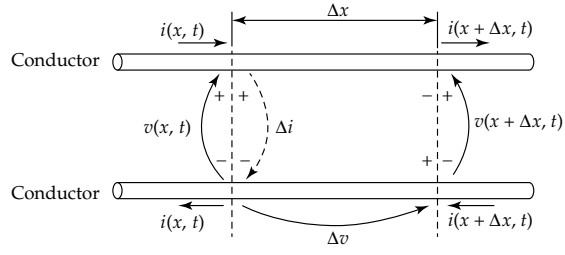
Therefore, longitudinal displacements in an elastic bar may be described by the one-dimensional wave equation with wave speed $\sqrt{E/\rho}$.

6.2.4 Transmission-Line Equations

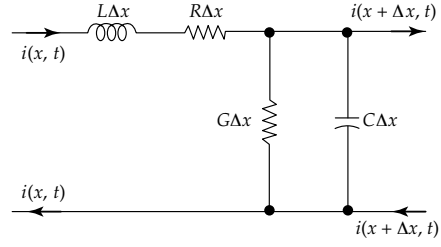
As a final example of wave motion, we shall derive the transmission-line equations. Electricity flows in the transmission line shown in Fig. 6.6, resulting in a current flow between conductors due to the capacitance and conductance between the conductors.

The cable also possesses both resistance and inductance resulting in voltage drops along the line. We shall choose the following symbols in our analysis:

$v(x, t)$ = voltage at any point along the line
$i(x, t)$ = current at any point along the line
R = resistance per meter
L = self-inductance per meter
C = capacitance per meter
G = conductance per meter



(a) Actual element



(b) Equivalent circuit

FIGURE 6.6 Element from a transmission line.

The voltage drop over the incremental length Δx at a particular instant (see Eqs. 1.4.3) is

$$\Delta v = v(x + \Delta x, t) - v(x, t) = -iR\,\Delta x - L\,\Delta x \frac{\partial i}{\partial t} \tag{6.2.30}$$

Dividing by Δx and taking the limit as $\Delta x \to 0$ yields the partial differential equation relating $v(x, t)$ and $i(x, t)$,

$$\frac{\partial v}{\partial x} + iR + L\frac{\partial i}{\partial t} = 0 \tag{6.2.31}$$

Now, let us find an expression for the change in the current over the length Δx. The current change is

$$\Delta i = i(x + \Delta x, t) - i(x, t) = -G\,\Delta x\, v - C\,\Delta x \frac{\partial v}{\partial t} \tag{6.2.32}$$

Again, dividing by Δx and taking the limit as $\Delta x \to 0$ gives a second equation,

$$\frac{\partial i}{\partial x} + vG + C\frac{\partial v}{\partial t} = 0 \tag{6.2.33}$$

Take the partial derivative of Eq. 6.2.31 with respect to x and of Eq. 6.2.33 with respect to t. Multiplying the second equation by L and subtracting the resulting two equations, using $\partial^2 i/\partial x\,\partial t = \partial^2 i/\partial t\,\partial x$, presents us with

$$\frac{\partial^2 v}{\partial x^2} + R\frac{\partial i}{\partial x} = LG\frac{\partial v}{\partial t} + LC\frac{\partial^2 v}{\partial t^2} \tag{6.2.34}$$

Then, substituting for $\partial i/\partial x$ from Eq. 6.2.33 results in an equation for $v(x, t)$ only. It is

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (LG + RC)\frac{\partial v}{\partial t} + RGv \tag{6.2.35}$$

Take the partial derivative of Eq. 6.2.31 with respect to t and multiply by C; take the partial derivative of Eq. 6.2.33 with respect to x, subtract the resulting two equations and substitute for $\partial v/\partial x$ from Eq. 6.2.31; there results

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (LG + RC)\frac{\partial i}{\partial t} + RGi \tag{6.2.36}$$

The two equations above are difficult to solve in the general form presented; two special cases are of interest. First, there are conditions under which the self-inductance, and leakage due to the conductance between conductors, are negligible; that is, $L \cong 0$, and $G \cong 0$. Then our equations become

$$\begin{aligned} \frac{\partial^2 v}{\partial x^2} &= RC\frac{\partial v}{\partial t} \\ \frac{\partial^2 i}{\partial x^2} &= RC\frac{\partial i}{\partial t} \end{aligned} \tag{6.2.37}$$

Second, for a condition of high frequency a time derivative increases the magnitude of a term*; that is, $\partial^2 i/\partial t^2 \gg \partial i/\partial t \gg i$. Thus, our general equations can be approximated by

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= \frac{1}{LC}\frac{\partial^2 v}{\partial x^2} \\ \frac{\partial^2 i}{\partial t^2} &= \frac{1}{LC}\frac{\partial^2 i}{\partial x^2} \end{aligned} \tag{6.2.38}$$

---

*As an example, consider the term $\sin(\omega t + x/L)$ where $\omega \gg 1$. Then

$$\frac{\partial}{\partial t}\left[\sin\left(\omega t + \frac{x}{L}\right)\right] = \omega\cos\left(\omega t + \frac{x}{L}\right)$$

We see that

$$\left|\omega\cos\left(\omega t + \frac{x}{L}\right)\right| \gg \left|\sin\left(\omega t + \frac{x}{L}\right)\right|$$

These latter two equations are wave equations with the units on $\sqrt{1/LC}$ of meters/second.

Although we shall not discuss any other wave phenomenon, it is well for the reader to be aware that sound waves, light waves, water waves, quantum-mechanical systems, and many other physical systems are described, at least in part, by a wave equation.

6.3 THE D'ALEMBERT SOLUTION OF THE WAVE EQUATION

It is possible to solve all the partial differential equations that we have derived in this chapter by a general method, the separation of variables. The wave equation can, however, be solved by a more special technique that will be presented in this section. It gives a quick look at the motion of a wave. We obtain a general solution to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{6.3.1}$$

by a proper transformation of variables. Introduce the new independent variables ξ(xi) and η(eta):

$$\xi = x - at, \qquad \eta = x + at \tag{6.3.2}$$

Then, using the chain rule we find that

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = -a\frac{\partial u}{\partial \xi} + a\frac{\partial u}{\partial \eta} \end{aligned} \tag{6.3.3}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial\left(\dfrac{\partial u}{\partial x}\right)}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial\left(\dfrac{\partial u}{\partial x}\right)}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial t^2} &= \frac{\partial\left(\dfrac{\partial u}{\partial t}\right)}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial\left(\dfrac{\partial u}{\partial t}\right)}{\partial \eta}\frac{\partial \eta}{\partial t} = a^2\frac{\partial^2 u}{\partial \xi^2} - 2a^2\frac{\partial^2 u}{\partial \xi\,\partial \eta} + a^2\frac{\partial^2 u}{\partial \eta^2} \end{aligned} \tag{6.3.4}$$

Substitute the expressions above into the wave equation to obtain

$$a^2\left[\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right] = a^2\left[\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi\,\partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right] \tag{6.3.5}$$

and there results

$$\frac{\partial^2 u}{\partial \xi\,\partial \eta} = 0 \tag{6.3.6}$$

Integration with respect to ξ gives

$$\frac{\partial u}{\partial \eta} = h(\eta) \tag{6.3.7}$$

where $h(\eta)$ is an arbitrary function of η (for an ordinary differential equation, this would be a constant). A second integration yields

$$u(\xi, \eta) = \int h(\eta)\, d\eta + g(\xi) \tag{6.3.8}$$

The integral is a function of η only and is replaced by $f(\eta)$, so the solution is

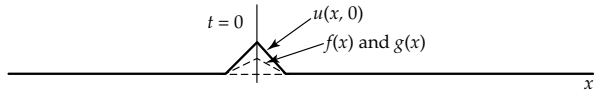
$$u(\xi, \eta) = g(\xi) + f(\eta) \tag{6.3.9}$$

or, equivalently,

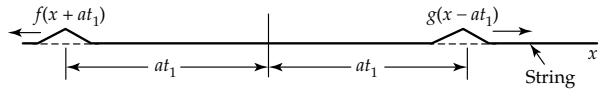
$$u(x,t) = g(x - at) + f(x + at) \tag{6.3.10}$$

This is the D'Alembert solution of the wave equation.

Inspection of the equation above shows the wave nature of the solution. Consider an infinite string, stretched from $-\infty$ to $+\infty$, with an initial displacement $u(x, 0) = g(x) + f(x)$, as shown in Fig. 6.7. At some later time $t = t_1$ the curves $g(x)$ and $f(x)$ will simply be displaced to the right and left, respectively, a distance at_1. The original deflection curves move without distortion at the speed of propagation a.



(a) Initial displacement



(b) Displacement after a time t_1

FIGURE 6.7 Traveling wave in a string.

To determine the form of the functions $g(x)$ and $f(x)$ when $u(x, 0)$ is given, we use the initial conditions. The term $\partial^2 u/\partial t^2$ in the wave equation demands that two bits of information be given at $t = 0$. Let us assume, for example, that the initial velocity is zero and that the initial displacement is given by

$$u(x,0) = f(x) + g(x) = \phi(x) \tag{6.3.11}$$

The velocity is

$$\frac{\partial u}{\partial t} = \frac{dg}{d\xi}\frac{\partial \xi}{\partial t} + \frac{df}{d\eta}\frac{\partial \eta}{\partial t} \tag{6.3.12}$$

At $t = 0$ this becomes (see Eqs. 6.3.2 and 6.3.10)

$$\frac{\partial u}{\partial t} = \frac{dg}{dx}(-a) + \frac{df}{dx}(a) = 0 \tag{6.3.13}$$

Hence, we have the requirement that

$$\frac{dg}{dx} = \frac{df}{dx} \tag{6.3.14}$$

which is integrated to provide us with

$$g = f + C \tag{6.3.15}$$

Inserting this in Eq. 6.3.11 gives

$$f(x) = \frac{\phi(x)}{2} - \frac{C}{2} \tag{6.3.16}$$

so that

$$g(x) = \frac{\phi(x)}{2} + \frac{C}{2} \tag{6.3.17}$$

Finally, replacing x in $f(x)$ with $x + at$ and x in $g(x)$ with $x - at$, there results the specific solution for the prescribed initial conditions,

$$u(x, t) = \frac{1}{2}\phi(x - at) + \frac{1}{2}\phi(x + at) \tag{6.3.18}$$

Our result shows that, for the infinite string, two initial conditions are necessary to determine the solution. A finite string will be discussed in the following section.

Example 6.1

Consider that the string in this article is given an initial velocity $\theta(x)$ and zero initial displacement. Determine the form of the solution.

Solution

The velocity is given by Eq. 6.3.12:

$$\frac{\partial u}{\partial t} = \frac{dg}{d\xi}\frac{\partial \xi}{\partial t} + \frac{df}{d\eta}\frac{\partial \eta}{\partial t}$$

At $t = 0$ this takes the form

$$\theta(x) = a\frac{df}{dx} - a\frac{dg}{dx}$$

This is integrated to yield

$$f - g = \frac{1}{a}\int_0^x \theta(s)\,ds + C$$

where s is a dummy variable of integration. The initial displacement is zero, giving

$$u(x, 0) = f(x) + g(x) = 0$$

or,

$$f(x) = -g(x)$$

The constant of integration C is thus evaluated to be

$$C = 2f(0) = -2g(0)$$

Combining this with the relation above results in

$$f(x) = \frac{1}{2a}\int_0^x \theta(s)\,ds + f(0)$$

$$g(x) = -\frac{1}{2a}\int_0^x \theta(s)\,ds + g(0)$$

Returning to Eq. 6.3.10, we can obtain the solution $u(x, t)$ using the forms above for $f(x)$ and $g(x)$ simply by replacing x by the appropriate quantity. We then have the solution

$$\begin{aligned} u(x,t) &= \frac{1}{2a}\left[\int_0^{x+at} \theta(s)ds - \int_0^{x-at} \theta(s)ds\right] \\ &= \frac{1}{2a}\left[\int_0^{x+at} \theta(s)\,ds + \int_{x-at}^{0} \theta(s)\,ds\right] \\ &= \frac{1}{2a}\int_{x-at}^{x+at} \theta(s)\,ds \end{aligned}$$

For a given $\theta(x)$ this expression would provide us with the solution.

Example 6.2

An infinite string is subjected to the initial displacement

$$\phi(x) = \frac{0.02}{1 + 9x^2}$$

Find an expression for the subsequent motion of the string if it is released from rest. The tension is 20 N and the mass per unit length is 5×10^{-4} kg/m. Also, sketch the solution for $t = 0$, $t = 0.002$ s, and $t = 0.01$ s.

Solution

The motion is given by the solution of this section. Equation 6.3.18 gives it as

$$u(x,t) = \frac{1}{2}\frac{0.02}{1 + 9(x - at)^2} + \frac{1}{2}\frac{0.02}{1 + 9(x + at)^2}$$

The wave speed a is given by

$$a = \sqrt{\frac{P}{m}}$$

$$= \sqrt{\frac{20}{5 \times 10^{-4}}} = 200 \text{ m/s}$$

The solution is then

$$u(x,t) = \frac{0.01}{1 + 9(x - 200t)^2} + \frac{0.01}{1 + 9(x + 200t)^2}$$

The sketches are presented in Fig. 6.8.

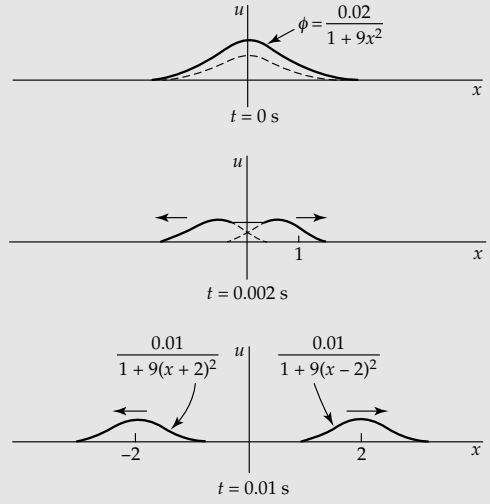


FIGURE 6.8

6.4 SEPARATION OF VARIABLES

We shall now present a powerful technique used to solve many of the partial differential equations encountered in physical applications in which the domains of interest are finite. It is the method of *separation of variables.* Even though it has limitations, it is a widely used technique. It involves the idea of reducing a more difficult problem to several simpler problems; here, we shall reduce a partial differential equation to several ordinary differential equations for which we already have methods of solution. Then, hopefully, by satisfying the initial and boundary conditions, a solution to the partial differential equation can be found.

To illustrate the details of the method, let us use the mathematical description of a finite string of length L that is fixed at both ends and is released from rest with an initial displacement. The motion of the string is described by the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{6.4.1}$$

We shall, as usual, consider the wave speed a to be a constant. The boundary conditions of fixed ends may be written as

$$u(0, t) = 0 \tag{6.4.2}$$

and

$$u(L, t) = 0 \tag{6.4.3}$$

Since the string is released from rest, the initial velocity is zero; hence,

$$\frac{\partial u}{\partial t}(x, 0) = 0 \tag{6.4.4}$$

The initial displacement will be denoted by $f(x)$; we then have

$$u(x, 0) = f(x) \tag{6.4.5}$$

We assume that the solution of our problem can be written in the separated form

$$u(x, t) = X(x)T(t) \tag{6.4.6}$$

that is, the x variable separates from the t variable. Substitution of this relationship into Eq. 6.4.1 yields

$$X(x)T''(t) = a^2 X''(x)T(t) \tag{6.4.7}$$

where the primes denote differentiation with respect to the associated independent variable. Rewriting Eq. 6.4.7 results in

$$\frac{T''}{a^2 T} = \frac{X''}{X} \tag{6.4.8}$$

The left side of this equation is a function of t only and the right side is a function of x only. Thus, as we vary t holding x fixed, the right side cannot change; this means that $T''(t)/a^2T(t)$ must be the same for all t. As we vary x holding t fixed the left side must not change. Thus, the quantity $X''(x)/X(x)$ must be the same for all x. Therefore, both sides must equal the same *constant* value μ (mu) sometimes called the *separation* constant. Equation 6.4.8 may then be written as two equations:

$$T'' - \mu a^2 T = 0 \tag{6.4.9}$$

$$X'' - \mu X = 0 \tag{6.4.10}$$

We note at this point that we have *separated* the *variables* and reduced a *partial differential equation* to two *ordinary differential equations*. If the boundary conditions can be

satisfied, then we have succeeded with our separation of variables. We shall assume that we need to consider μ only as a real number. Thus, we are left with the three cases;

$$\begin{gathered}\mu > 0 \\ \mu = 0 \\ \mu < 0\end{gathered} \tag{6.4.11}$$

For any nonzero value of μ, we know that the solutions of these second-order ordinary differential equations are of the form e^{mt} and e^{nx}, respectively (see Section 1.5). The characteristic equations are

$$m^2 - \mu a^2 = 0 \tag{6.4.12}$$

$$n^2 - \mu = 0 \tag{6.4.13}$$

The roots are

$$m_1 = a\sqrt{\mu}, \quad m_2 = -a\sqrt{\mu} \tag{6.4.14}$$

$$n_1 = \sqrt{\mu}, \qquad n_2 = -\sqrt{\mu} \tag{6.4.15}$$

The resulting solutions are

$$T(t) = c_1 e^{\sqrt{\mu}\, at} + c_2 e^{-\sqrt{\mu}\, at} \tag{6.4.16}$$

and

$$X(x) = c_3 e^{\sqrt{\mu}\, x} + c_4 e^{-\sqrt{\mu}\, x} \tag{6.4.17}$$

Now, consider the three cases, $\mu > 0$, $\mu = 0$, and $\mu < 0$. For $\mu > 0$, we have the result that $\sqrt{\mu}$ is a real number and the general solution is

$$u(x, t) = T(t)X(x) = (c_1 e^{\sqrt{\mu}\, at} + c_2 e^{-\sqrt{\mu}\, at})(c_3 e^{\sqrt{\mu}\, x} + c_4 e^{-\sqrt{\mu}\, x}) \tag{6.4.18}$$

which is a decaying or growing exponential. The derivative of Eq. 6.4.18 with respect to time would yield the velocity and it, too, would be growing or decaying with respect to time. This, of course, means that the kinetic energy of an element of the string would be increasing or decreasing in time, as would the total kinetic energy. However, energy remains constant; therefore, this solution violates the basic law of physical conservation of energy. The solution also does not give the desired wave motion and the boundary and initial conditions cannot be satisfied; thus, we cannot have $\mu > 0$. Similar arguments prohibit the use of $\mu = 0$. Hence, we are left with $\mu < 0$; for simplicity, let

$$\sqrt{\mu} = i\beta \tag{6.4.19}$$

where β is a real number and i is $\sqrt{-1}$. For this case, Eq. 6.4.16 becomes

$$T(t) = c_1 e^{i\beta at} + c_2 e^{-i\beta at} \tag{6.4.20}$$

and Eq. 6.4.17 becomes

$$X(x) = c_3 e^{i\beta x} + c_4 e^{-i\beta x} \tag{6.4.21}$$

Using the relation

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{6.4.22}$$

Eqs. 6.4.20 and 6.4.21 may be rewritten as

$$T(t) = A\sin\beta at + B\cos\beta at \tag{6.4.23}$$

and

$$X(x) = C\sin\beta x + D\cos\beta x \tag{6.4.24}$$

where A, B, C, and D are new constants. The relation of the new constants in terms of the constants c_1, c_2, c_3, and c_4 is left as an exercise.

Now that we have solutions to Eqs. 6.4.9 and 6.4.10 that are periodic in time and space, let us attempt to satisfy the boundary conditions and initial conditions given in Eqs. 6.4.2 through 6.4.5. Our solution thus far is

$$u(x, t) = (A\sin\beta at + B\cos\beta at)(C\sin\beta x + D\cos\beta x) \tag{6.4.25}$$

The boundary condition $u(0, t) = 0$ states that u is zero for all t at $x = 0$; that is,

$$u(0, t) = (A\sin\beta at + B\cos\beta at)D = 0 \tag{6.4.26}$$

The only way this is possible is to have $D = 0$. Hence, we are left with

$$u(x, t) = (A\sin\beta at + B\cos\beta at)C\sin\beta x \tag{6.4.27}$$

The boundary condition $u(L, t) = 0$ states that u is zero for all t at $x = L$; this is expressed as

$$u(L, t) = (A\sin\beta at + B\cos\beta at)C\sin\beta L \tag{6.4.28}$$

This is possible if

$$\sin\beta L = 0 \tag{6.4.29}$$

For this to be true, we must have

$$\beta L = n\pi, \quad n = 1, 2, 3, \cdots \tag{6.4.30}$$

or $\beta = n\pi/L$; the quantity β is called an *eigenvalue.* When the β is substituted back into $\sin\beta x$, the function $\sin n\pi x/L$ is called an *eigenfunction.* Each eigenvalue corresponding to a particular value of n produces a unique eigenfunction. Note that the $n = 0$ eigenvalue ($\mu = 0$) has already been eliminated as a possible solution, so it is not included here. The solution given in Eq. 6.4.27 may now be written as

$$u(x, t) = \left(A\sin\frac{n\pi at}{L} + B\cos\frac{n\pi at}{L}\right)C\sin\frac{n\pi x}{L} \tag{6.4.31}$$

For simplicity, let us make the substitutions

$$AC = a_n, \quad BC = b_n \tag{6.4.32}$$

since each value of n may require different constants. Equation 6.4.31 is then

$$u_n(x,t) = \left(a_n \sin\frac{n\pi at}{L} + b_n \cos\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L} \tag{6.4.33}$$

where the subscript n has been added to $u(x, t)$ to allow for a different function for each value of n.

For our vibrating string, each value of n results in harmonic motion of the string with frequency $na/2L$ cycles per second (hertz). For $n = 1$ the *fundamental mode* results, and for $n > 1$ *overtones* result; see Fig. 6.9. *Nodes* are those points of the string which do not move. The velocity $\partial u_n/\partial t$ is then

$$\frac{\partial u_n}{\partial t} = \frac{n\pi a}{L}\left(a_n \cos\frac{n\pi at}{L} - b_n \sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L} \tag{6.4.34}$$

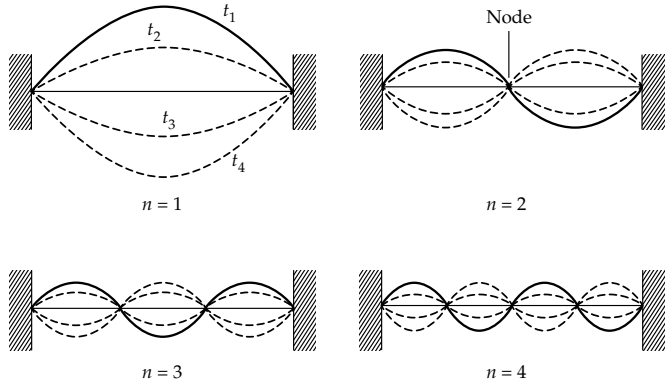


FIGURE 6.9 Harmonic motion. The solution at various values of time t is as shown.

Thus, to satisfy b.c. (6.4.4),

$$\frac{\partial u_n}{\partial t}(x,0) = \frac{n\pi a}{L} a_n \sin\frac{n\pi x}{L} = 0 \tag{6.4.35}$$

for all x, we must have $a_n = 0$. We are now left with

$$u_n(x,t) = b_n \cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L} \tag{6.4.36}$$

If we are to solve our problem, we must satisfy boundary condition (6.4.5),

$$u_n(x,0) = f(x) \tag{6.4.37}$$

But, unless $f(x)$ is a multiple of $\sin n\pi x/L$, no one value of n will satisfy Eq. 6.4.37. How do we then satisfy the boundary condition $u(x, 0) = f(x)$ if $f(x)$ is not a sine function?

Equation 6.4.36 is a solution of Eq. 6.4.1 and satisfies Eqs. 6.4.2 through 6.4.4 for all n, $n = 1, 2, 3, \cdots$. Hence, any linear combination of any of the solutions

$$u_n(x,t) = b_n \cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \cdots \tag{6.4.38}$$

is also a solution, since the describing equation is linear and superposition is allowed. If we assume that for the most general function $f(x)$ we need to consider all values of n, then we should try

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}b_n\cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L} \tag{6.4.39}$$

For the i.c. (6.4.5), we then have

$$u(x,0)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}=f(x) \tag{6.4.40}$$

If constants b_n can be determined to satisfy Eq. 6.4.40, then we have a solution anywhere that the sum in Eq. 6.4.39 converges. The series in Eq. 6.4.40 is a *Fourier sine series.* It was presented in Section 1.10, but the essential features will be repeated here.

To find the b_n's, multiply the right side of Eq. 6.4.40 by sin $m\pi x/L$ to give

$$\sin\frac{m\pi x}{L}\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}=f(x)\sin\frac{m\pi x}{L} \tag{6.4.41}$$

Now integrate both sides of Eq. 6.4.41 from $x=0$ to $x=L$. We may take sin $m\pi x/L$ inside the sum, since it is a constant as far as the summation is concerned. The integral and the summation may be switched if the series converges properly. This may be done for most functions of interest in physical applications. Thus, we have

$$\sum_{n=1}^{\infty}b_n\int_0^L\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx=\int_0^L f(x)\sin\frac{m\pi x}{L}dx \tag{6.4.42}$$

With the use of trigonometric identities we can verify* that

$$\int_0^L\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx=\begin{cases}0 & \text{if } m\neq n\\ \dfrac{L}{2} & \text{if } m=n\end{cases} \tag{6.4.43}$$

Hence, Eq. 6.4.42 gives us

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \tag{6.4.44}$$

if $f(x)$ may be expressed by

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L} \tag{6.4.45}$$

Equation 6.4.45 gives the Fourier sine series representation of $f(x)$ with the coefficients given by Eq. 6.4.44. Examples will illustrate the use of the above equations for particular functions $f(x)$.

---

*Use the trigonometric identities $\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$ and $\sin^2\alpha=\frac{1}{2}-\frac{1}{2}\cos 2\alpha$.

Example 6.3

A tight string 2 m long with $a = 30$ m/s is initially at rest but is given an initial velocity of 300 sin $4\pi x$ from its equilibrium position. Determine the maximum displacement at the $x = \frac{1}{8}$ m location of the string.

Solution

We assume that the solution to the describing differential equation

$$\frac{\partial^2 u}{\partial t^2} = 900 \frac{\partial^2 u}{\partial x^2}$$

can be separated as

$$u(x, t) = T(t)X(x)$$

Following the procedure of the previous section, we substitute into the describing equation to obtain

$$\frac{1}{900}\frac{T''}{T} = \frac{X''}{X} = -\beta^2$$

where we have chosen the separation constant to be $-\beta^2$ so that an oscillatory motion will result. The two ordinary differential equations that result are

$$T'' + 900\beta^2 T = 0$$

$$X'' + \beta^2 X = 0$$

The general solutions to the equations above are

$$T(t) = A \sin 30\beta t + B \cos 30\beta t$$

$$X(x) = C \sin \beta x + D \cos \beta x$$

The solution for $u(x, t)$ is then

$$u(x, t) = (A \sin 30\beta t + B \cos 30\beta t)(C \sin \beta x + D \cos \beta x)$$

The end at $x = 0$ remains motionless; that is, $u(0, t) = 0$. Hence,

$$u(0, t) = (A \sin 30\beta t + B \cos 30\beta t)(0 + D) = 0$$

Thus, $D = 0$. The initial displacement $u(x, 0) = 0$. Hence,

$$u(x, 0) = (0 + B)C \sin \beta x = 0$$

Thus, $B = 0$. The solution reduces to

$$u(x, t) = AC \sin 30\beta t \sin \beta x$$

The initial velocity $\partial u/\partial t$ is given as 300 sin $4\pi x$. We then have, at $t = 0$,

$$\frac{\partial u}{\partial t} = 30\beta AC \sin \beta x = 300 \sin 4\pi x$$

This gives

$$\beta = 4\pi, \quad AC = \frac{300}{30(4\pi)} = \frac{2.5}{\pi}$$

The solution for the displacement is finally

$$u(x, t) = \frac{2.5}{\pi} \sin 120\pi t \sin 4\pi x$$

We have not imposed the condition that the end at $x = 2$ m is motionless. Insert $x = 2$ in the expression above and it is obvious that this boundary condition is satisfied; thus we have found an acceptable solution.

The maximum displacement at $x = 1/8$ m occurs when sin $120\pi t = 1$. Thus, the maximum displacement is

$$u_{\max} = \frac{2.5}{\pi} \text{ m}$$

Note that we did not find it necessary to use the general expression given by Eq. 6.4.39. We could have, but it would have required more work to obtain a solution. This happened because the initial condition was given as a sine function. Any other function would require the more general form given by Eq. 6.4.39.

Example 6.4

Determine several coefficients in the series solution for $u(x, t)$ for the vibrating string if

$$f(x) = \begin{cases} 0.1x & 0 \le x \le 1 \\ 0.2 - 0.1x & 1 < x \le 2 \end{cases}$$

The string is 2 m long. Use the boundary and initial conditions of Section 6.4.

Solution

The solution for the displacement of the string is given by Eq. 6.4.39. It is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi at}{2} \sin \frac{n\pi x}{2}$$

where we have used $L = 2$ m. The coefficients b_n are related to the initial displacement $f(x)$ by Eq. 6.4.44,

$$b_n = \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} dx$$

Substituting for $f(x)$ results in

$$b_n = 0.1\int_0^1 x\sin\frac{n\pi x}{2}dx + 0.1\int_1^2 (2-x)\sin\frac{n\pi x}{2}dx$$

Performing the integrations (integration by parts* is required) gives

$$b_n = 0.1\left[-\frac{2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^1 + 0.1\left[-\frac{4}{n\pi}\cos\frac{n\pi x}{2} + \frac{2x}{n\pi}\cos\frac{n\pi x}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_1^2$$

By being careful in reducing this result, we have

$$b_n = \frac{0.8}{\pi^2 n^2}\sin\frac{n\pi}{2}$$

This gives several b_n's as

$$b_1 = \frac{0.8}{\pi^2},\quad b_2 = 0,\quad b_3 = -\frac{0.8}{9\pi^2},\quad b_4 = 0,\quad b_5 = \frac{0.8}{25\pi^2}$$

The solution is, finally,

$$u(x,t) = \frac{0.8}{\pi^2}\left[\cos\frac{\pi at}{2}\sin\frac{\pi x}{2} - \frac{1}{9}\cos\frac{3\pi at}{2}\sin\frac{3\pi x}{2} + \frac{1}{25}\cos\frac{5\pi at}{2}\sin\frac{5\pi x}{2} + \cdots\right]$$

We see that the amplitude of each term is getting smaller and smaller. A good approximation results if we keep several terms (say five) and simply ignore the rest. This, in fact, was done before the advent of the computer. With the computer many more terms can be retained, with accurate numbers resulting from the calculations. A computer plot of the solution above is shown in Fig. 6.10 for a = 100 m/s. One hundred terms were retained.

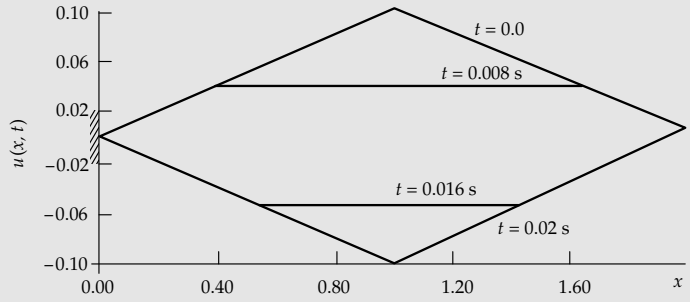


FIGURE 6.10

*We shall integrate $\int_0^\pi x\sin x\,dx$ by parts. Let $u = x$ and $dv = \sin x\,dx$. Then $du = dx$ and $v = -\cos x$. The integral is then $\int_0^\pi x\sin x\,dx = -x\cos x\Big|_0^\pi + \int_0^\pi \cos x\,dx = \pi$.

Example 6.5

A tight string, π m long and fixed at both ends, is given an initial displacement $f(x)$ and an initial velocity $g(x)$. Find an expression for $u(x, t)$.

Solution

We follow the steps of Section 6.4 and find the general solution to be

$$u(x, t) = (A \sin \beta at + B \cos \beta at)(C \sin \beta x + D \cos \beta x)$$

Using the b.c. that the left end is fixed, that is, $u(0, t) = 0$, we have $D = 0$. We also have the b.c. that $u(\pi, t) = 0$, giving

$$0 = (A \sin \beta at + B \cos \beta at) C \sin \beta \pi.$$

If we let $C = 0$, a trivial solution results, $u(x, t) = 0$. Thus, we must let

$$\beta \pi = n\pi$$

or $\beta = n$, an integer. The general solution is then

$$u_n(x, t) = (a_n \sin nat + b_n \cos nat) \sin nx$$

where the subscript n on $u_n(x, t)$ allows for a different $u(x, t)$ for each value of n. The most general $u(x, t)$ is then found by superposing all of the $u_n(x, t)$; that is,

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) = \sum_{n=1}^{\infty} (a_n \sin nat + b_n \cos nat) \sin nx \tag{1}$$

Now, to satisfy the initial displacement, we require that

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx = f(x)$$

Multiply by sin mx and integrate from 0 to π. Using the results indicated in Eq. 6.4.43, we have

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \tag{2}$$

Next, to satisfy the initial velocity, we must have

$$\frac{\partial u}{\partial t}(x, 0) = \sum_{n=1}^{\infty} a_n an \sin nx = g(x)$$

Again, multiply by sin mx and integrate from 0 to π. Then

$$a_n = \frac{2}{an\pi} \int_0^{\pi} g(x) \sin nx \, dx \tag{3}$$

Our solution is now complete. It is given by Eq. 1 with the b_n provided by Eq. 2 and the a_n by Eq. 3. If $f(x)$ and $g(x)$ were specified numerical values for each b_n and a_n would result.

Example 6.6

A tight string, π m long, is fixed at the left end but the right end moves, with displacement 0.2 sin 15t. Find $u(x, t)$ if the wave speed is 30 m/s and state the initial conditions if a solution using separation of variables is to be possible.

Solution
Separation of variables leads to the general solution as

$$u(x, t) = (A \sin 30\beta t + B \cos 30\beta t)(C \sin \beta x + D \cos \beta x)$$

The left end is fixed, requiring that $u(0, t) = 0$. Hence, $D = 0$. The right end moves with the displacement 0.2 sin 15t; that is,

$$u(\pi, t) = 0.2 \sin 15t = (A \sin 30\beta t + B \cos 30\beta t) C \sin \beta\pi$$

This can be satisfied if we let

$$B = 0, \qquad \beta = \frac{1}{2}, \qquad AC = 0.2$$

The resulting solution for $u(x, t)$ is

$$u(x, t) = 0.2 \sin 15t \sin \frac{x}{2}$$

The initial displacement $u(x, 0)$ must be zero and the initial velocity must be

$$\frac{\partial u}{\partial t}(x, 0) = 3 \sin \frac{x}{2}$$

Any other set of initial conditions would not allow a solution using separation of variables.

Example 6.7

A tight string is fixed at both ends. A forcing function (this could be due to wind blowing over a wire), applied normal to the string, is given by $\mathcal{F}(t) = Km \sin \omega t$ kilograms per meter of length. Show that resonance occurs whenever $\omega = an\pi/L$.

Solution
The forcing function $\mathcal{F}(t)$ multiplied by the distance Δx can be added to the right-hand side of Eq. 6.2.8. Dividing by $m\Delta x$ results in

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + K \sin \omega t$$

where $a^2 = P/m$. This is a nonhomogeneous partial differential equation, since the last term does not contain the dependent variable $u(x, t)$. As with ordinary differential

equations that are linear, we can find a particular solution and add it to the solution of the associated homogeneous equation to form the general solution.

We assume that the effect of the forcing function will be to produce a displacement having the same frequency as the forcing function, as was the case with lumped systems. This suggests that the particular solution has the form

$$u_p(x, t) = X(x) \sin \omega t$$

Substituting this into the partial differential equation gives

$$a^2 X'' \sin \omega t = -X\omega^2 \sin \omega t + K \sin \omega t$$

The sin ωt divides out and we are left with the ordinary differential equation

$$X'' + \frac{\omega^2}{a^2} X = K$$

The general solution to this nonhomogeneous differential equation is (see Section 1.8)

$$X(x) = c_1 \sin \frac{\omega}{a} x + c_2 \cos \frac{\omega}{a} x + \frac{Ka^2}{\omega^2}$$

We will force this solution to satisfy the end conditions that apply to the string. Hence,

$$X(0) = 0 = c_2 + \frac{Ka^2}{\omega^2}$$

$$X(L) = 0 = c_1 \sin \frac{\omega L}{a} + c_2 \cos \frac{\omega L}{a} + \frac{Ka^2}{\omega^2}$$

The equations above give

$$c_2 = -\frac{Ka^2}{\omega^2}, \qquad c_1 = \frac{\dfrac{Ka^2}{\omega^2}\left(\cos \dfrac{\omega L}{a} - 1\right)}{\sin(\omega L/a)}$$

The particular solution is then

$$u_p(x, t) = \frac{Ka^2}{\omega^2}\left[\frac{\cos \dfrac{\omega L}{a} - 1}{\sin(\omega L/a)} \sin \frac{\omega x}{a} - \cos \frac{\omega x}{a} + 1\right] \sin \omega t$$

The amplitude of the above becomes infinite whenever $\sin \omega L/a = 0$ and $\cos \omega L/a = -1$. This occurs whenever

$$\frac{\omega L}{a} = (2n - 1)\pi$$

Hence, if the input frequency is such that

$$\omega = \frac{(2n-1)\pi a}{L}, \qquad n = 1, 2, 3, \cdots$$

the amplitude of the resulting motion becomes infinitely large. This equals the natural frequency corresponding to the fundamental mode or one of the significant overtones of the string, depending on the value of n. Thus, we see that a number of input frequencies can lead to resonance in the string. This is true of all phenomena modeled by the wave equation. Recall that we have neglected any type of damping.

6.5 DIFFUSION

Another class of physical problems exists that is characterized by diffusion equations. Diffusion may be likened to a spreading, smearing, or mixing. A physical system that has a high concentration of variable ϕ in volume A and a low concentration of ϕ in volume B may tend to *diffuse* so that the concentrations in A and B approach equality. This phenomenon is exhibited by the tendency of a body toward a uniform temperature. One of the most common diffusion processes that is encountered is the transfer of energy in the form of heat.

From thermodynamics we learn that heat is thermal energy in transit. It may be transmitted by conduction (when two bodies are in contact), by convection (when a body is in contact with a liquid or a gas), and by radiation (when energy is transmitted by energy waves). We shall consider the first of these mechanisms in some detail. Experimental observations have been organized to permit us to make the following two statements:

1. Heat flows in the direction of decreasing temperature.
2. The rate at which energy in the form of heat is transferred through an area is proportional to the area and to the temperature gradient normal to the area.

These statements must be expressed analytically. The heat flux through an area A oriented normal to the x axis is

$$Q = -KA\frac{\partial T}{\partial x} \tag{6.5.1}$$

where Q (watts per second, W/s) is the heat flux, $\partial T/\partial x$ is the temperature gradient normal to A, and K (W/m · s · K) is a constant of proportionality called the *thermal conductivity.* The minus sign is present since heat is transferred in the direction opposite the temperature gradient.

The energy (usually called internal energy) gained or lost by a body of mass m that undergoes a uniform temperature change ΔT is expressed as

$$\Delta E = Cm\,\Delta T \tag{6.5.2}$$

where ΔE (W) the energy change of the body and C (W/kg · K) is a constant of proportionality called the *specific heat.*

Conservation of energy is a fundamental law of nature. We shall use this law to make an energy balance on the element in Fig. 6.11. The density ρ of the element will be used to determine its mass, namely,

$$m = \rho \Delta x \Delta y \Delta z \tag{6.5.3}$$

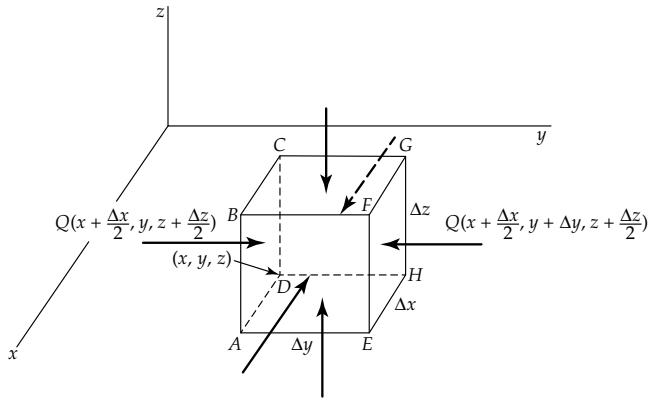


FIGURE 6.11 Element of mass.

By energy balance we mean that the net energy flowing into the element in time Δt must equal the increase in energy in the element in Δt. For simplicity, we shall assume that there are no sources inside the element. Equation 6.5.2 gives the change in energy in the element as

$$\Delta E = Cm\Delta T = C\rho \Delta x \Delta y \Delta z \Delta T \tag{6.5.4}$$

The energy that flows into the element through face $ABCD$ in Δt is, by Eq. 6.5.1,

$$\Delta E_{ABCD} = Q_{ABCD}\Delta t = -K \Delta x \Delta z \left.\frac{\partial T}{\partial y}\right|_{\substack{x+\Delta x/2 \\ y \\ z+\Delta z/2}} \Delta t \tag{6.5.5}$$

where we have approximated the temperature derivative by the value at the center of the face. The flow into the element through face $EFGH$ is

$$\Delta E_{EFGH} = K \Delta x \Delta z \left.\frac{\partial T}{\partial y}\right|_{\substack{x+\Delta x/2 \\ y+\Delta y \\ z+\Delta z/2}} \Delta t \tag{6.5.6}$$

Similar expressions are found for the other four faces. The energy balance then provides us with

$$\Delta E = \Delta E_{ABCD} + \Delta E_{EFGH} + \Delta E_{ADHE} + \Delta E_{BCGF} + \Delta E_{DHGC} + \Delta E_{BFEA} \tag{6.5.7}$$

or, using Eqs. 3.5.5, 3.5.6, and their counterparts for the x and z directions,

$$C\rho\,\Delta x\,\Delta y\,\Delta z\,\Delta T = K\,\Delta x\,\Delta z\left(\left.\frac{\partial T}{\partial y}\right|_{\substack{x+\Delta x/2\\ y+\Delta y\\ z+\Delta z/2}} - \left.\frac{\partial T}{\partial y}\right|_{\substack{x+\Delta x/2\\ y\\ z+\Delta z/2}}\right)\Delta t$$
$$+K\,\Delta y\,\Delta z\left(\left.\frac{\partial T}{\partial x}\right|_{\substack{x+\Delta x\\ y+\Delta y/2\\ z+\Delta z/2}} - \left.\frac{\partial T}{\partial x}\right|_{\substack{x\\ y+\Delta y/2\\ z+\Delta z/2}}\right)\Delta t$$
$$+K\,\Delta x\,\Delta y\left(\left.\frac{\partial T}{\partial z}\right|_{\substack{x+\Delta x/2\\ y+\Delta y/2\\ z+\Delta z}} - \left.\frac{\partial T}{\partial z}\right|_{\substack{x+\Delta x/2\\ y+\Delta y/2\\ z}}\right)\Delta t \tag{6.5.8}$$

Both sides of the equation are divided by $C\rho\ \Delta x\ \Delta y\ \Delta z\ \Delta t$, then let $\Delta x \to 0$, $\Delta y \to 0$, $\Delta z \to 0$, $\Delta t \to 0$; there results

$$\frac{\partial T}{\partial t} = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] \tag{6.5.9}$$

where $k = K/C\rho$ is called the *thermal diffusivity* and is assumed constant. It has dimensions of square meters per second (m^2/s). Equation 6.5.9 is a *diffusion equation*

Two special cases of the diffusion equation are of particular interest. A number of situations involve time and only one coordinate, say x, as in a long, slender rod with insulated sides. The *one-dimensional heat equation* then results. It is given by

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{6.5.10}$$

which is a parabolic equation.

In some situations $\partial T/\partial t$ is zero and we have a steady-state condition; then we no longer have a diffusion equation, but the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{6.5.11}$$

This equation is known as *Laplace's equation*. It is sometimes written in the shorthand form

$$\nabla^2 T = 0 \tag{6.5.12}$$

If the temperature depends only on two coordinates x and y, as in a thin rectangular plate, an elliptic equation is encountered,

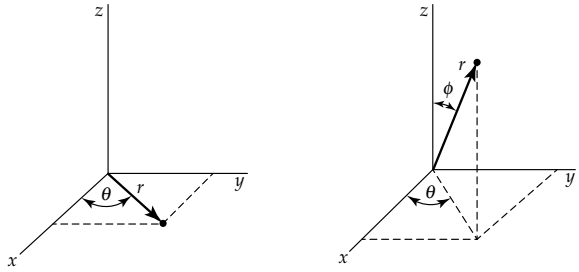
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{6.5.13}$$

Cylindrical or spherical coordinates (see Fig. 6.12) should be used in certain geometries. It is then convienient to express $\nabla^2 T$ in cylindrical coordinates as

$$\nabla^2 T = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{6.5.14}$$

and in spherical coordinates as

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 T}{\partial\theta^2} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial T}{\partial\phi}\right) \tag{6.5.15}$$



(a) Cylindrical coordinates (b) Spherical coordinates

FIGURE 6.12 Cylindrical and spherical coordinates.

Our discussion of heat transfer has included heat conduction only. Radiative and convective forms of heat transfer would necessarily lead to other partial differential equations. We have also assumed no heat sources in the volume of interest, and have assumed the conductivity K to be constant. Finally, the specification of boundary and initial conditions would make our problem statement complete. These will be reserved for the following section in which a solution to the diffusion equation is presented.

6.6 SOLUTION OF THE DIFFUSION EQUATION

This section will be devoted to a solution of the diffusion equation developed in Section 6.5. Recall that the diffusion equation is

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{6.6.1}$$

Heat transfer will again be used to illustrate this very important phenomenon. The procedure developed for the wave equation will be used, but the solution will be quite different, owing to the presence of the first derivative with respect to time rather than the second derivative. This requires only one initial condition instead of the two required by the wave equation. We shall illustrate the solution technique with three specific situations.

6.6.1 A Long, Insulated Rod with Ends at Fixed Temperatures

A long rod, shown in Fig. 6.13, is subjected to an initial temperature distribution along its axis; the rod is insulated on the lateral surface, and the ends of the rod are kept at the same constant temperature.* The insulation prevents heat flux in the radial direction;

*We shall choose the temperature of the ends in the illustration to be 0°C. Note, however, that both ends could be held at any temperature T_0. Since it is necessary to have the ends maintained at zero, we would simply define a new variable $\theta = T - T_0$, so that $\theta = 0$ at both ends. We would then find a solution for $\theta(x, t)$ with the desired temperature given by $T(x, t) = \theta(x, t) + T_0$.

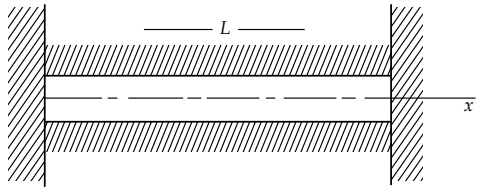


FIGURE 6.13 Heated rod.

hence, the temperature will depend on the x coordinate only. The describing equation is then the one-dimensional heat equation, given by Eq. 6.5.10, as

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \tag{6.6.2}$$

We shall choose to hold the ends at $T = 0°$. These boundary conditions are expressed as

$$T(0, t) = 0, \qquad T(L, t) = 0 \tag{6.6.3}$$

Let the initial temperature distribution be represented by

$$T(x, 0) = f(x) \tag{6.6.4}$$

We assume that the variables separate; that is,

$$T(x, t) = \theta(t)X(x) \tag{6.6.5}$$

Substitution of Eq. 6.6.5 into 6.6.2 yields

$$\theta' X = k\theta X'' \tag{6.6.6}$$

where $\theta' = d\theta/dt$ and $X'' = d^2X/dx^2$. This is rearranged as

$$\frac{\theta'}{k\theta} = \frac{X''}{X} \tag{6.6.7}$$

Since the left side is a function of t only and the right side is a function of x only, we set Eq. 6.6.7 equal to a constant λ (lambda). This gives

$$\theta' - \lambda k\theta = 0 \tag{6.6.8}$$

and

$$X'' - \lambda X = 0 \tag{6.6.9}$$

The solution of Eq. 6.6.8 is of the form

$$\theta(t) = c_1 e^{\lambda k t} \tag{6.6.10}$$

Equation 6.6.9 yields the solution

$$X(x) = c_2 e^{\sqrt{\lambda}x} + c_3 e^{-\sqrt{\lambda}x} \tag{6.6.11}$$

Again, we must decide whether

$$\lambda > 0, \qquad \lambda = 0, \qquad \lambda < 0 \tag{6.6.12}$$

For $\lambda > 0$, Eq. 6.6.10 shows that the solution has a nearly infinite temperature at large t due to exponential growth; of course, this is not physically possible. For $\lambda = 0$, the solution would be independent of time. Again our physical intuition tells us this is not expected. Therefore, we are left with $\lambda < 0$. If we can satisfy the boundary conditions, then we have found a solution. Let

$$\beta^2 = -\lambda \tag{6.6.13}$$

so that

$$\beta^2 > 0 \tag{6.6.14}$$

The solutions, Eqs. 6.6.10 and 6.6.11, may then be written as

$$\theta(t) = Ae^{-\beta^2 kt} \tag{6.6.15}$$

and

$$X(x) = B\sin\beta x + C\cos\beta x \tag{6.6.16}$$

where A, B, and C are arbitrary constants to be determined. Therefore, our solution is

$$T(x,t) = Ae^{-\beta^2 kt}\left[B\sin\beta x + C\cos\beta x\right] \tag{6.6.17}$$

The first condition of Eq. 6.6.3 implies that

$$C = 0 \tag{6.6.18}$$

Therefore, our solution reduces to

$$T(x,t) = De^{-\beta^2 kt}\sin\beta x \tag{6.6.19}$$

where $D = A \cdot B$. The second boundary condition of Eq. 6.6.3 requires that

$$\sin\beta L = 0 \tag{6.6.20}$$

This is satisfied if

$$\beta L = n\pi, \quad \text{or} \quad \beta = n\pi/L, \quad n = 1, 2, 3, \ldots \tag{6.6.21}$$

The constant β is the eigenvalue, and the function $\sin n\pi x/L$ is the eigenfunction. The solution is now

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty} D_n e^{-kn^2\pi^2 t/L^2}\sin\frac{n\pi x}{L} \tag{6.6.22}$$

The initial condition, (6.6.4), may be satisfied at $t = 0$ if

$$T(x,0) = f(x) = \sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{L} \tag{6.6.23}$$

that is, if $f(x)$ can be expanded in a Fourier sine series. If such is the case, the coefficients will be given by (refer to Eqs. 6.4.42–6.4.44)

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \tag{6.6.24}$$

and the separation-of-variables technique is successful.

It should be noted again that all solutions of partial differential equations cannot be found by separation of variables; in fact, it is only a very special set of boundary conditions that allows us to separate the variables. For example, Eq. 6.6.20 would obviously not be useful in satisfying the boundary condition $T(L, t) = 20t$. Separation of variables would then be futile. Numerical methods could be used to find a solution, or other analytical techniques not covered in this book would be necessary.

Example 6.8

A long copper rod with insulated lateral surfaces has its left end maintained at a temperature of 0°C and its right end, at $x = 2$ m, maintained at 100°C. Determine the temperature as a function of x and t if the initial condition is given by

$$T(x,0) = f(x) = \begin{cases} 100x & 0 < x < 1 \\ 100 & 1 < x < 2 \end{cases}$$

The thermal diffusivity for copper is $k = 1.14 \times 10^{-4}$ m^2/s.

Solution

We again assume the variables separate as

$$T(x, t) = \theta(t)X(x)$$

with the resulting equation,

$$\frac{1}{k}\frac{\theta'}{\theta} = \frac{X''}{X} = \lambda$$

In this problem the eigenvalue $\lambda = 0$ will play an important role. The solution for $\lambda = 0$ is

$$\theta(t) = C_1, \qquad X(x) = A_1x + B_1$$

resulting in

$$T(x, t) = C_1(A_1x + B_1)$$

To satisfy the two end conditions $T(0, t) = 0$ and $T(2, t) = 100$, it is necessary to require $B_1 = 0$ and $A_1C_1 = 50$. Then

$$T(x, t) = 50x \tag{1}$$

This solution is, of course, independent of time, but we will find it quite useful.

Now, we return to the case that allows for exponential decay of temperature, namely $\lambda = -\beta^2$. For this eigenvalue see Eq. 6.6.17 the solution is

$$T(x, t) = Ae^{-\beta^2kt}\left[B\sin\beta x + C\cos\beta x\right] \tag{2}$$

We can superimpose the above two solutions, since Eq. 6.6.2 is linear, and obtain the more general solution

$$T(x, t) = 50x + Ae^{-\beta^2kt}\left[B\sin\beta x + C\cos\beta x\right]$$

Now let us satisfy the boundary conditions. The left-end condition $T(0, t) = 0$ demands that $C = 0$. The right-end condition demands that

$$100 = 100 + A \cdot Be^{-\beta^2 kt} \sin \beta L$$

This requires that $\sin \beta L = 0$, which occurs whenever

$$\beta L = n\pi \quad \text{or} \quad \beta = n\pi/L, \qquad n = 1, 2, 3, \cdots$$

The general solution is then

$$T(x, t) = 50x + \sum_{n=1}^{\infty} D_n e^{-n^2\pi^2 kt/4} \sin \frac{n\pi x}{2}$$

using $L = 2$. Note that this satisfies the describing equation (6.6.2) and the two boundary conditions. Finally, it must satisfy the initial condition

$$f(x) = 50x + \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{2}$$

We see that if the function $[f(x) - 50x]$ can be expanded in a Fourier sine series, then the solution will be complete. The Fourier coefficients are

$$\begin{aligned} D_n &= \frac{2}{L}\int_0^L [f(x) - 50x] \sin \frac{n\pi x}{L}\, dx \\ &= \frac{2}{2}\int_0^1 (100x - 50x) \sin \frac{n\pi x}{2}\, dx + \frac{2}{2}\int_1^2 (100 - 50x) \sin \frac{n\pi x}{2}\, dx \\ &= 50\left[-\frac{2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right]_0^1 - \frac{200}{n\pi} \cos \frac{n\pi x}{2} \Bigg|_1^2 \\ &\quad - 50\left[-\frac{2x}{n\pi} \cos \frac{n\pi x}{2} + \frac{4}{n^2\pi^2} \sin \frac{n\pi x}{2} \right]_1^2 \\ &= \frac{400}{n^2\pi^2} \sin \frac{n\pi}{2} \end{aligned}$$

The solution is, using $k = 1.14 \times 10^{-4}$ m^2/s for copper,

$$T(x, t) = 50x + \sum_{n=1}^{\infty} \frac{40.5}{n^2} \sin \frac{n\pi}{2} e^{-2.81\times 10^{-4} n^2 t} \sin \frac{n\pi x}{2}$$

Note that the time t is measured in seconds.

6.6.2 A Long, Totally Insulated Rod

The lateral sides of the long rod are again insulated so that heat transfer occurs only in the x direction along the rod. The temperature in the rod is described by the one-dimensional heat equation

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{6.6.25}$$

For this problem, we have an initial temperature distribution given by

$$T(x,0) = f(x) \tag{6.6.26}$$

Since the rod is totally insulated, the heat flux across the end faces is zero. This condition gives, with the use of Eq. 6.5.1,

$$\frac{\partial T}{\partial x}(0,t) = 0, \qquad \frac{\partial T}{\partial x}(L,t) = 0 \tag{6.6.27}$$

We assume that the variables separate,

$$T(x,t) = \theta(t)X(x) \tag{6.6.28}$$

Substitute into Eq. 6.6.25, to obtain

$$\frac{\theta'}{k\theta} = \frac{X''}{X} = -\beta^2 \tag{6.6.29}$$

where $-\beta^2$ is a negative real number. Equation 6.6.29 gives

$$\theta' = -\beta^2 k\theta \tag{6.6.30}$$

and

$$X'' + \beta^2 X = 0 \tag{6.6.31}$$

The equations have solutions in the form

$$\theta(t) = Ae^{-\beta^2 kt} \tag{6.6.32}$$

and

$$X(x) = B\sin\beta x + C\cos\beta x \tag{6.6.33}$$

The first boundary condition of (6.6.27) implies that $B = 0$, and the second requires that

$$\frac{\partial X}{\partial x}(L) = -C\beta\sin\beta L = 0 \tag{6.6.34}$$

This can be satisfied if we set

$$\sin\beta L = 0 \tag{6.6.35}$$

hence, the eigenvalues are

$$\beta = \frac{n\pi}{L}, \quad n = 0, 1, 2, \cdots \tag{6.6.36}$$

Thus, the independent solutions are of the form

$$T_n(x,t) = a_n e^{-n^2\pi^2 kt/L^2}\cos\frac{n\pi x}{L} \tag{6.6.37}$$

where the constant a_n replaces AC. The general solution, which hopefully will satisfy the remaining initial condition, is then

$$T(x,t) = \sum_{n=0}^{\infty} a_n e^{-(n^2\pi^2 k/L^2)t} \cos\frac{n\pi x}{L} \tag{6.6.38}$$

Note that we retain the $\beta = 0$ eigenvalue in the series.

The initial condition is given by Eq. 6.6.26. It demands that

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} \tag{6.6.39}$$

Using trigonometric identities we can show that

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n \neq 0 \\ L & m = n = 0 \end{cases} \tag{6.6.40}$$

Multiply both sides of Eq. 6.6.39 by $\cos m\pi x/L$ and integrate from 0 to L. We then have*

$$a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \tag{6.6.41}$$

The solution is finally

$$T(x,t) = \sum_{n=0}^{\infty} a_n e^{-(n^2\pi^2 k/L^2)t} \cos\frac{n\pi x}{L} \tag{6.6.42}$$

Thus, the temperature distribution can be determined provided that $f(x)$ can be expanded in a Fourier cosine series.

Example 6.9

A long, laterally insulated stainless steel rod has heat generation occurring within the rod at the constant rate of 4140 $W/m^3 \cdot s$. The right end is insulated and the left end is maintained at 0°C. Find an expression for $T(x, t)$ if the initial temperature distribution is

$$T(x,0) = f(x) = \begin{cases} 100x & 0 < x < 1 \\ 200 - 100x & 1 < x < 2 \end{cases}$$

for the 2-m-long, 0.1-m-diameter rod. Use the specific heat C = 460 J/kg · °C, $\rho = 7820$ kg/m^3, and $k = 3.86 \times 10^{-6}$ m^2/s.

*Note that it is often the practice to define a_0 as $a_0 = (2/L)\int_0^L f(x)dx$ and then to write the solution as $T(x,t) = a_0/2 + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2 kt/L^2}\cos(n\pi x/L)$. This was done in Section 1.10. Both methods are, of course, equivalent.

Solution
To find the appropriate describing equation, we must account for the heat generated in the infinitesimal element of Fig. 6.11. To Eq. 6.5.7 we would add a heat-generation term,

$$\phi(x, y, z, t)\,\Delta x\,\Delta y\,\Delta z\,\Delta t$$

where $\phi(x, y, z, t)$ is the amount of heat generated per volume per unit time. The one-dimensional heat equation would then take the form

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \frac{\phi}{\rho C}$$

For the present example the describing equation is

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \frac{4140}{7890 \cdot 460}$$

This nonhomogeneous, partial differential equation is solved by finding a particular solution and adding it to the solution of the homogeneous equation

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

The solution of the homogeneous equation is (see Eqs. 6.6.32 and 6.6.33)

$$T(x, t) = Ae^{-\beta^2 kt}\left[B\sin\beta x + C\cos\beta x\right]$$

The left-end boundary condition is $T(0, t) = 0$, resulting in $C = 0$. The insulated right end requires that $\partial T/\partial x\,(L, t) = 0$. This results in

$$\cos\beta L = 0$$

Thus, the quantity βL must equal $\pi/2, 3\pi/2, 5\pi/2, \cdots$. This is accomplished by using

$$\beta = \frac{(2n-1)\pi}{2L}, \quad n = 1, 2, 3, \cdots$$

The homogeneous solution is, then, using $k = 3.86 \times 10^{-6}$ and $L = 2$,

$$T(x, t) = \sum_{n=1}^{\infty} D_n e^{-2.38\times10^{-6}(2n-1)^2 t}\sin\left(\frac{2n-1}{4}\pi x\right)$$

To find the particular solution, we note that the generation of heat is independent of time. Since the homogeneous solution decays to zero with time, we anticipate that the heat-generation term will lead to a steady-state temperature distribution. Thus, we assume the particular solution to be independent of time, that is,

$$T_p(x, t) = g(x)$$

Substitute this into the describing equation, to obtain

$$0 = 3.86 \times 10^{-6} g'' + 1.15 \times 10^{-3}$$

The solution to this ordinary differential equation is

$$g(x) = -149x^2 + c_1 x + c_2$$

This solution must also satisfy the boundary condition at the left end, yielding $c_2 = 0$ and the boundary condition at the right end ($g' = 0$), giving $c_1 = 596$. The complete solution, which must now satisfy the initial condition, is

$$T(x,t) = -149x^2 + 596x + \sum_{n=1}^{\infty} D_n e^{-2.38\times10^{-6}(2n-1)^2 t} \sin\left(\frac{2n-1}{4}\pi x\right)$$

To find the unknown coefficients D_n we use the initial condition, which states that

$$f(x) = -149x^2 + 596x + \sum_{n=1}^{\infty} D_n \sin\left(\frac{2n-1}{4}\pi x\right)$$

The coefficients are then

$$D_n = \frac{2}{2}\int_0^2 [f(x) + 149x^2 - 596x]\sin\left(\frac{2n-1}{4}\pi x\right)dx$$

$$= \int_0^1 (149x^2 - 496x)\sin\left(\frac{2n-1}{4}\pi x\right)dx$$

$$+ \int_1^2 (149x^2 - 696x + 200)\sin\left(\frac{2n-1}{4}\pi x\right)dx$$

The integrals can be integrated by parts providing a complete solution.

6.6.3 Two-Dimensional Heat Conduction in a Long, Rectangular Bar

A long, rectangular bar is bounded by the planes $x = 0$, $x = a$, $y = 0$, and, $y = b$. These faces are kept at $T = 0$°C, as shown by the cross section in Fig. 6.14. The bar is heated so that

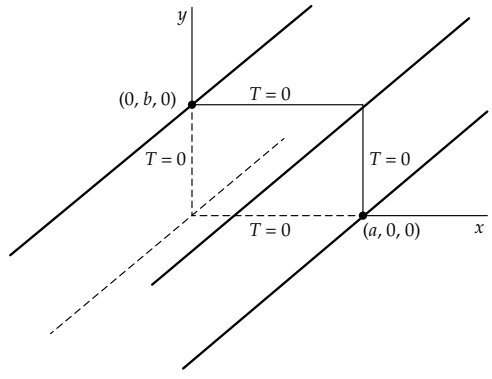


FIGURE 6.14 Cross section of a rectangular bar.

the variation in the z direction may be neglected. Thus, the variation of temperature in the bar is described by

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{6.6.43}$$

The initial temperature distribution in the bar is given by

$$T(x, y, 0) = f(x, y) \tag{6.6.44}$$

We want to find an expression for $T(x, y, t)$. Hence, we assume that

$$T(x, y, t) = X(x)Y(y)\theta(t) \tag{6.6.45}$$

After Eq. 6.6.45 is substituted into Eq. 6.6.43, we find that

$$XY\theta' = k(X''Y\theta + XY''\theta) \tag{6.6.46}$$

Equation 6.6.46 may be rewritten as

$$\frac{X''}{X} = \frac{\theta'}{k\theta} - \frac{Y''}{Y} \tag{6.6.47}$$

Since the left-hand side of Eq. 6.6.47 is a function of x only and the right side is a function of t and y, we may assume that both sides equal the constant value $-\lambda$. (With experience we now anticipate the minus sign.) Therefore, we have

$$X'' + \lambda X = 0 \tag{6.6.48}$$

and

$$\frac{Y''}{Y} = \frac{\theta'}{k\theta} + \lambda \tag{6.6.49}$$

We use the same argument on Eq. 6.6.49 and set it equal to a constant $-\mu$. That is,

$$\frac{Y''}{Y} = \frac{\theta'}{k\theta} + \lambda = -\mu \tag{6.6.50}$$

This yields the two differential equations

$$Y'' + \mu Y = 0 \tag{6.6.51}$$

and

$$\theta' + (\lambda + \mu)k\theta = 0 \tag{6.6.52}$$

The boundary conditions on $X(x)$ are

$$X(0) = 0, \qquad X(a) = 0 \tag{6.6.53}$$

since the temperature is zero at $x = 0$ and $x = a$. Consequently, the solution of Eq. 6.6.48,

$$X(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x \tag{6.6.54}$$

reduces to

$$X(x) = A \sin \frac{n\pi x}{a} \tag{6.6.55}$$

where we have used

$$\lambda = \frac{n^2\pi^2}{a^2}, \qquad n = 1, 2, 3, \cdots \tag{6.6.56}$$

Similarly, the solution to Eq. 6.6.51 reduces to

$$Y(y) = C \sin \frac{m\pi y}{b} \tag{6.6.57}$$

where we have employed

$$\mu = \frac{m^2\pi^2}{b^2}, \qquad m = 1, 2, 3 \cdots \tag{6.6.58}$$

With the use of Eqs. 6.6.56 and 6.6.58 we find the solution of Eq. 6.6.52 to be

$$\theta(t) = De^{-\pi^2 k(n^2/a^2 + m^2/b^2)t} \tag{6.6.59}$$

Equations 6.6.55, 6.6.57 and 6.6.59 may be combined to give

$$T_{mn}(x, y, t) = A_{mn} e^{-\pi^2 k(n^2/a^2 + m^2/b^2)t} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \tag{6.6.60}$$

where the constant a_{mn} replaces ACD. The most general solution is then obtained by superposition, namely,

$$T(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} T_{mn} \tag{6.6.61}$$

and we have

$$T(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} e^{-\pi^2 k(n^2/a^2 + m^2/b^2)t} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b} \tag{6.6.62}$$

This is a solution if coefficients a_{mn} can be determined so that

$$T(x, y, 0) = f(x, y) = \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} a_{mn} \sin \frac{n\pi x}{a} \right] \sin \frac{m\pi y}{b} \tag{6.6.63}$$

We make the grouping indicated by the brackets in Eq. 6.6.63. Thus, for a given x in the range $(0, a)$, we have a Fourier series in y. [For a given x, $f(x, y)$ is a function of y only.] Therefore, the term in the brackets is the constant b_n in the Fourier sine series. Hence,

$$\begin{aligned} \sum_{n=1}^{\infty} a_{mn} \sin \frac{n\pi x}{a} &= \frac{2}{b} \int_0^b f(x, y) \sin \frac{m\pi y}{b} dy \\ &= F_m(x) \end{aligned} \tag{6.6.64}$$

The right-hand side of Eq. 6.6.64 is a series of functions of x, one for each $m = 1, 2, 3, \cdots$. Thus, Eq. 6.6.64 is a Fourier sine series for $F_m(x)$. Therefore, we have

$$a_{mn} = \frac{2}{a}\int_0^a F_m(x)\sin\frac{n\pi x}{a}dx \tag{6.6.65}$$

Substitution of Eq. 6.6.64 into Eq. 6.6.65 yields

$$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi y}{b}\sin\frac{n\pi x}{a}dydx \tag{6.6.66}$$

The solution of our problem is Eq. 6.6.62 with a_{mn} given by Eq. 6.6.66.

This problem is an example of an extension of the ideas that we have developed, to include three independent variables; the two-dimensional Fourier series representation was also utilized.

We have studied the major ideas used in the application of separation of variables to problems in rectangular coordinates; to find the solution it was, in general, necessary to expand the initial condition in a Fourier series. For other problems that are more conveniently formulated in cylindrical coordinates, we would find Bessel functions taking the place of Fourier series, and using spherical coordinates, Legendre polynomials would appear. Sections 6.7 and 6.8 will present the solutions to Laplace's equation in spherical coordinates and cylindrical coordinates, respectively.

Example 6.10

The edges of a thin plate are held at the temperatures shown in the sketch of Fig. 6.15. Determine the steady-state temperature distribution in the plate. Assume the large plate surfaces to be insulated.

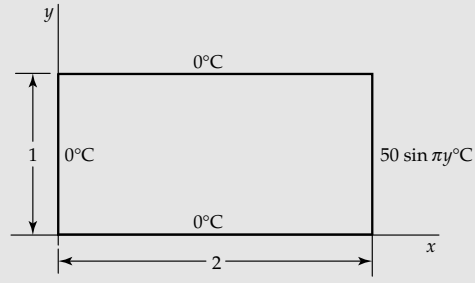


FIGURE 6.15

Solution

The describing equation is the heat equation

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

For the steady-state situation there is no variation of temperature with time; that is, $\partial T/\partial t = 0$. For a thin plate with insulated surfaces we have $\partial^2 T/\partial z^2 = 0$. Thus,

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

This is Laplace's equation. Let us assume that the variables separate; that is,

$$T(x, y) = X(x)Y(y)$$

Then substitute into the describing equation to obtain

$$\frac{X''}{X} = -\frac{Y''}{Y} = \beta^2$$

where we have chosen the separation constant to be positive to allow for a sinusoidal variation* with y. The ordinary differential equations that result are

$$X'' - \beta^2 X = 0$$
$$Y'' + \beta^2 Y = 0$$

The solutions are

$$X(x) = Ae^{\beta x} + Be^{-\beta x}$$
$$Y(y) = C\sin\beta y + D\cos\beta y$$

The solution for $T(x, y)$ is then

$$T(x, y) = (Ae^{\beta x} + Be^{-\beta x})(C\sin\beta y + D\cos\beta y)$$

Using $T(0, y) = 0$, $T(x, 0) = 0$, and $T(x, 1) = 0$ gives

$$0 = A + B$$
$$0 = D$$
$$0 = \sin\beta$$

The final boundary condition is

$$T(2, y) = 50\sin\pi y = (Ae^{2\beta} + Be^{-2\beta})C\sin\beta y$$

From this condition we have

$$\beta = \pi$$
$$50 = C(Ae^{2\beta} + Be^{-2\beta})$$

From the equations above we can solve for the constants. We have

$$B = -A, \qquad AC = \frac{50}{e^{2\pi} - e^{-2\pi}} = 0.0934$$

Finally, the expression for $T(x, y)$ is

$$T(x, y) = 0.0934(e^{\pi x} - e^{-\pi x})\sin\pi y$$

Note that the expression above for the temperature is independent of the material properties; it is a steady-state solution.

*If the right-hand edge were held at a constant temperature we would also choose the separation constant so that cos βy and sin βy appear. This would allow a Fourier series to satisfy the edge condition.

6.7 ELECTRIC POTENTIAL ABOUT A SPHERICAL SURFACE

Consider that a spherical surface is maintained at an electrical potential V. The potential depends only on ϕ and is given by the function $f(\phi)$. The equation that describes the potential in the region on either side of the spherical surface is Laplace's equation (6.5.15), written in spherical coordinates (shown in Fig. 6.12) as

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial\phi}\right)=0 \tag{6.7.1}$$

Obviously, one boundary condition requires that

$$V(r_0,\phi)=f(\phi) \tag{6.7.2}$$

The fact that a potential exists on the spherical surface of finite radius should not lead to a potential at infinite distances from the sphere; hence, we set

$$V(\infty,\phi)=0 \tag{6.7.3}$$

We follow the usual procedure of separating variables; that is, assume that

$$V(r,\phi)=R(r)\Phi(\phi) \tag{6.7.4}$$

This leads to the equations

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=-\frac{1}{\Phi\sin\theta}\frac{d}{d\phi}(\Phi\sin\phi)=\mu \tag{6.7.5}$$

which can be written as, letting $\cos\phi=x$, so that $\Phi=\Phi(x)$,

$$\begin{aligned} r^2R''+2rR'-\mu R&=0\\ (1-x^2)\Phi''-2x\Phi+\mu\Phi&=0 \end{aligned} \tag{6.7.6}$$

The first of these is recognized as Cauchy's equation (see Section 1.11) and has the solution

$$R(r)=c_1r^{-1/2+\sqrt{\mu+1/4}}+c_2r^{-1/2-\sqrt{\mu+1/4}} \tag{6.7.7}$$

This is put in better form by letting $-\frac{1}{2}+\sqrt{\mu+\frac{1}{4}}=n$. Then

$$R(r)=c_1r^n+\frac{c_2}{r^{n+1}} \tag{6.7.8}$$

The equation for Φ becomes Legendre's equation (see Section 2.3),

$$(1-x^2)\Phi''-2x\Phi'+n(n+1)\Phi=0 \tag{6.7.9}$$

where n must be a positive integer for a proper solution to exist. The general solution to this equation is

$$\Phi(x)=c_3P_n(x)+c_4Q_n(x) \tag{6.7.10}$$

Since $Q_n(x) \to \infty$ as $x \to 1$ (see Eq. 2.3.19), we set $c_4 = 0$. This results in the following solution for $V(r, x)$:

$$V(r,x) = \sum_{n=0}^{\infty} V_n(r,x) = \sum_{n=0}^{\infty} [A_n r^n P_n(x) + B_n r^{-(n+1)} P_n(x)] \tag{6.7.11}$$

Let us first consider points inside the spherical surface. The constants $B_n = 0$ if a finite potential is to exist at $r = 0$. We are left with

$$V(r,x) = \sum_{n=0}^{\infty} A_n r^n P_n(x) \tag{6.7.12}$$

This equation must satisfy the boundary condition

$$V(r_0,x) = f(x) = \sum_{n=0}^{\infty} A_n r_0^n P_n(x) \tag{6.7.13}$$

The unknown coefficients A_n are found by using the property

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \begin{cases} 0 & m \neq n \\ \dfrac{2}{2n+1} & m = n \end{cases} \tag{6.7.14}$$

Multiply both sides of Eq. 6.7.12 by $P_m(x)dx$ and integrate from –1 to 1. This gives

$$A_n = \frac{2n+1}{2r_0^n} \int_{-1}^{1} f(x) P_n(x) dx \tag{6.7.15}$$

For a prescribed $f(\phi)$, using $\cos \phi = x$, Eq. 6.7.12 provides us with the solution for interior points with the constants A_n given by Eq. 6.7.15.

For exterior points we require that $A_n = 0$ in Eq. 6.7.11, so the solution is bounded as $x \to \infty$. This leaves the solution

$$V(r,x) = \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(x) \tag{6.7.16}$$

This equation must also satisfy the boundary condition

$$f(x) = \sum_{n=0}^{\infty} B_n r_0^{-(n+1)} P_n(x) \tag{6.7.17}$$

Using the property (6.7.14), the B_n's are given by

$$B_n = \frac{2n+1}{2} r_0^{n+1} \int_{-1}^{1} f(x) P_n(x) dx \tag{6.7.18}$$

If $f(x)$ is a constant we must evaluate $\int_{-1}^{1} P_n(x)\,dx$. Using Eq. 2.3.15 we can show that

$$\int_{-1}^{1} P_0(x) dx = 2, \quad \int_{-1}^{1} P_n(x)\,dx = 0, \qquad n = 1, 2, 3, \cdots \tag{6.7.19}$$

An example will illustrate the application of this presentation for a specific $f(x)$.

Example 6.11

Find the electric potential inside a spherical surface of radius r_0 if the hemispherical surface when $\pi > \phi > \pi/2$ is maintained at a constant potential V_0 and the hemispherical surface when $\pi/2 > \phi > 0$ is maintained at zero potential.

Solution

Inside the sphere of radius r_0, the solution is

$$V(r,x) = \sum_{n=0}^{\infty} A_n r^n P_n(x)$$

where $x = \cos\phi$. The coefficients A_n are given by Eq. 6.7.15,

$$\begin{aligned} A_n &= \frac{2n+1}{2r_0^n}\int_{-1}^{1} f(x)P_n(x)\,dx \\ &= \frac{2n+1}{2r_0^n}\left[\int_{-1}^{0} V_0 P_n(x)\,dx + \int_{0}^{1} 0\cdot P_n(x)\,dx\right] \\ &= \frac{2n+1}{2r_0^n}V_0\int_{-1}^{0} P_n(x)\,dx \end{aligned}$$

where we have used $V = V_0$ for $\pi > \phi > \pi/2$ and $V = 0$ for $\pi/2 > \phi > 0$. Several A_n's can be evaluated, to give (see Eq. 2.3.15)

$$A_0 = \frac{V_0}{2}, \quad A_1 = -\frac{3V_0}{4r_0}, \quad A_2 = 0, \quad A_3 = \frac{7V_0}{16r_0^3}, \quad A_4 = 0, \quad A_5 = -\frac{11V_0}{32r_0^5}$$

This provides us with the solution, letting $\cos\phi = x$,

$$\begin{aligned} V(r,\phi) &= A_0P_0 + A_1rP_1 + A_2r^2P_2 + \cdots \\ &= V_0\left[\frac{1}{2} - \frac{3}{4}\frac{r}{r_0}\cos\phi + \frac{7}{16}\left(\frac{r}{r_0}\right)^3 P_3(\cos\phi) - \frac{11}{32}\left(\frac{r}{r_0}\right)^5 P_5(\cos\phi) + \cdots\right] \end{aligned}$$

where the Legendre polynomials are given by Eqs. 2.3.15. Note that the expression above could be used to give a reasonable approximation to the temperature in a solid sphere if the hemispheres are maintained at T_0 and zero degrees, respectively, since Laplace's equation also describes the temperature distribution in a solid body.

6.8 HEAT TRANSFER IN A CYLINDRICAL BODY

Boundary-value problems involving a boundary condition applied to a circular cylindrical surface are encountered quite often in physical situations. The solution of such problems invariably involve Bessel functions, which were introduced in

Section 2.5. We shall use the problem of finding the steady-state temperature distribution in the cylinder shown in Fig. 6.16 as an example. Other exercises are included in the Problems.

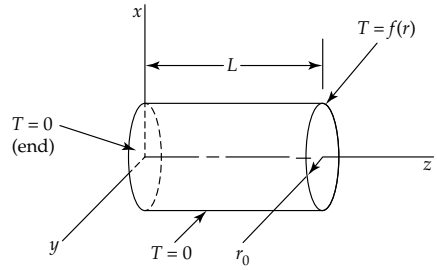


FIGURE 6.16 Circular cylinder with boundary conditions.

The partial differential equation describing the phenomenon illustrated in Fig. 6.16 is

$$\frac{\partial T}{\partial t} = k\nabla^2 T \tag{6.8.1}$$

where we have assumed constant material properties. For a steady-state situation using cylindrical coordinates (see Eq. 6.5.14), this becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{6.8.2}$$

where, considering the boundary conditions shown in the figure, we have assumed the temperature to be independent of θ. We assume a separated solution of the form

$$T(r, z) = R(r)Z(z) \tag{6.8.3}$$

which leads to the equations

$$\frac{1}{R}\left(R'' + \frac{1}{r}R'\right) = -\frac{Z''}{Z} = -\mu^2 \tag{6.8.4}$$

where a negative sign is chosen on the separation constant since we anticipate an exponential variation with z. We are thus confronted with solving the two ordinary differential equations

$$R'' + \frac{1}{2}R' + \mu^2 R = 0 \tag{6.8.5}$$

$$Z'' - \mu^2 Z = 0 \tag{6.8.6}$$

The solution to Eq. 6.8.6 is simply

$$Z(z) = c_1 e^{\mu z} + c_2 e^{-\mu z} \tag{6.8.7}$$

for $\mu > 0$; for $\mu = 0$, it is

$$Z(z) = c_5 z + c_6 \tag{6.8.8}$$

This solution may or may not be of use. We note that Eq. 6.8.5 is close to being Bessel's equation (2.5.1) with $\lambda = 0$. By substituting $x = \mu r$, Eq. 6.8.5 becomes

$$x^2R'' + xR' + x^2R = 0 \tag{6.8.9}$$

which is Bessel's equation with $\lambda = 0$. It possesses the general solution

$$R(x) = c_3J_0(x) + c_4Y_0(x) \tag{6.8.10}$$

where $J_0(x)$ and $Y_0(x)$ are Bessel functions of the first and second kind, respectively. We know (see Fig. 2.5) that $Y_0(x)$ is singular at $x = 0$. (This corresponds to $r = 0$.) Hence, we require that $c_4 = 0$, and the solution to our problem is

$$T(r, z) = J_0(\mu r)[Ae^{\mu z} + Be^{-\mu z}] \tag{6.8.11}$$

The temperature on the surface at $z = 0$ is maintained at zero degrees. This gives $B = -A$ from the equation above. The temperature at $r = r_0$ is also maintained at zero degrees; that is,

$$T(r_0, z) = 0 = AJ_0(\mu r_0)[e^{\mu z} - e^{-\mu z}] \tag{6.8.12}$$

The Bessel function $J_0(\mu r_0)$ has infinitely many roots that allow the equation above to be satisfied; none of these roots equal zero; thus the $\mu = 0$ eigenvalue is not of use. Let the nth root be designated μ_n. Four such roots are shown in Fig. 2.4 and are given numerically in the Appendix.

Returning to Eq. 6.8.11, our solution is now

$$T(r, z) = \sum_{n=1}^{\infty} T_n(r, z) = \sum_{n=1}^{\infty} J_0(\mu_n r)A_n[e^{\mu_n z} - e^{-\mu_n z}] \tag{6.8.13}$$

This solution should allow the final end condition to be satisfied. It is

$$T(r, L) = f(r) = \sum_{n=1}^{\infty} A_nJ_0(\mu_n r)[e^{\mu_n L} - e^{-\mu_n L}] \tag{6.8.14}$$

We must now use the property that

$$\int_0^b xJ_j(\mu_n x)J_j(\mu_m x)\,dx = \begin{cases} 0 & n \neq m \\ \dfrac{b^2}{2}J_{j+1}^2(\mu_n b) & n = m \end{cases} \tag{6.8.15}$$

where the μ_n are the roots of the equation $J_j(\mu r_0) = 0$. This permits the coefficients A_n to be determined from, using $j = 0$,

$$A_n = \frac{2(e^{\mu_n L} - e^{-\mu_n L})^{-1}}{r_0^2J_1^2(\mu_n r_0)} \int_0^{r_0} rf(r)J_0(\mu_n r)\,dr \tag{6.8.16}$$

This completes the solution. For a specified $f(r)$ for the temperature on the right end, Eq. 6.8.13 gives the temperature at any interior point if the coefficients are evaluated using Eq. 6.8.16. This process will be illustrated with an example.

Example 6.12

Determine the steady-state temperature distribution in a 2-unit-long, 4-unit-diameter circular cylinder with one end maintained at 0°C, the other end at $100r$ °C, and the lateral surface insulated.

Solution

Following the solution procedure outlined in the previous section, the solution is

$$T(r, z) = J_0(\mu r)[Ae^{\mu z} + Be^{-\mu z}]$$

The temperature at the base where $z = 0$ is zero. Thus, $B = -A$ and

$$T(r, z) = AJ_0(\mu r)[e^{\mu z} + e^{-\mu z}]$$

On the lateral surface where $r = 2$, the heat transfer is zero, requiring that

$$\frac{\partial T}{\partial r}(2, z) = 0 = AJ_0'(2\mu)[e^{\mu z} - e^{-\mu z}]$$

or

$$J_0'(2\mu) = 0$$

There are infinitely many values of μ that provide this condition, the first of which is $\mu = 0$. Let the nth one be μ_n, the eigenvalue. The solution corresponding to this eigenvalue is

$$T_n(r, z) = A_n J_0(\mu_n r)[e^{\mu_n z} - e^{-\mu_n z}]$$

for $\mu_n > 0$; for $\mu_1 = 0$, the solution is, using Eq. 6.8.8,

$$T_1(r, z) = A_1 z$$

The general solution is then found by superimposing all the individual solutions, resulting in

$$T(r, z) = \sum_{n=1}^{\infty} T_n(r, z) = A_1 z + \sum_{n=2}^{\infty} A_n J_0(\mu_n r)[e^{\mu_n z} - e^{-\mu_n z}]$$

The remaining boundary condition is that the end at $z = 2$ is maintained at $100r$ °C, that is,

$$T(r, 2) = 100r = 2A_1 + \sum_{n=2}^{\infty} A_n J_0(\mu_n r)[e^{2\mu_n} - e^{-2\mu_n}]$$

We must be careful, however, and not assume that the A_n in this series are given by Eq. 6.8.16; they are not, since the roots μ_n are not to the equation $J_0(\mu r_0) = 0$, but to $J'_0(\mu r_0) = 0$. The property analogous to Eq. 6.8.15 takes the form

$$\int_0^{r_0} x J_j(\mu_n x) J_j(\mu_m x)\, dx = \begin{cases} 0 & n \neq m \\ \dfrac{\mu_n^2 r_0^2 - j^2}{2\mu_n^2} J_j^2(\mu_n r_0) & n = m \end{cases}$$

whenever μ_n are the roots of $J_j'(\mu r_0) = 0$. The coefficients A_n are then given by, using $j = 0$,

$$A_n = \frac{2(e^{2\mu_n} - e^{-2\mu_n})^{-1}}{r_0^2 J_0^2(\mu_n r_0)} \int_0^{r_0} r f(r) J_0(\mu_n r)\, dr$$

where $f(r) = 100r$. For the first root, $\mu_1 = 0$, the coefficient is

$$A_1 = \frac{2}{r_0^2} \int_0^{r_0} r f(r)\, dr$$

Some of the coefficients are, using $\mu_1 = 0$, $\mu_2 = 1.916$, $\mu_3 = 3.508$

$$A_1 = \frac{2}{2^2} \int_0^2 r(100r)\, dr = \frac{400}{3}$$

$$A_2 = \frac{2(e^{3.832} - e^{-3.832})^{-1}}{2^2 \times 0.403^2} \int_0^2 r(100r) J_0(1.916r)\, dr$$

$$= 6.68 \int_0^2 r^2 J_0(1.916r)\, dr = 0.951 \int_0^{3.832} x^2 J_0(x)\, dx$$

$$A_3 = \frac{2(e^{7.016} - e^{-7.016})^{-1}}{2^2 \times 0.300^2} \int_0^2 r(100r) J_0(3.508r)\, dr$$

$$= 0.501 \int_0^2 r^2 J_0(3.508r)\, dr = 0.0117 \int_0^{7.016} x^2 J_0(x)\, dx$$

The integrals above could be easily evaluated by use of a computer integration scheme. Such a scheme will be presented in Chapter 8. The solution is then

$$T(r, z) = \frac{400}{3} z + A_2 J_0(1.916r)[e^{1.916z} - e^{-1.916z}] + A_3 J_0(3.508r)[e^{3.508z} - e^{-3.508z}] + \cdots$$

6.9 GRAVITATIONAL POTENTIAL

There are a number of physical situations that are modeled by Laplace's equation. We shall choose the force of attraction of particles to demonstrate its derivation. The law of gravitation states that a lumped mass m located at the point (X, Y, Z) attracts a unit mass located at the point (x, y, z) (see Fig. 6.17), with a force directed along the line connecting the two points with magnitude given by

$$F = -\frac{km}{r^2} \tag{6.9.1}$$

where K is a positive constant and the negative sign indicates that the force acts toward the mass m. The distance between the two points is provided by the expression

$$r = \sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2} \tag{6.9.2}$$

positive being from Q to P.

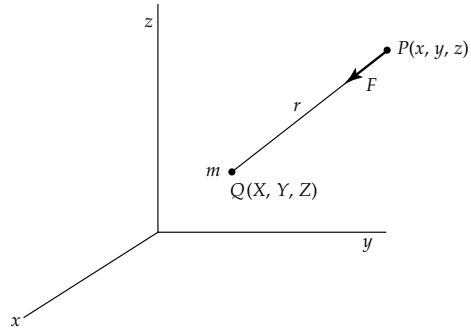


FIGURE 6.17 Gravitational attraction.

A gravitational potential ϕ can be defined as

$$\phi = \frac{km}{r} \tag{6.9.3}$$

This allows the force F acting on a unit mass at P due to a mass at Q to be related to ϕ by the equation

$$\begin{aligned} F &= \frac{\partial \phi}{\partial r} \\ &= -\frac{km}{r^2} \end{aligned} \tag{6.9.4}$$

Now, let the mass m be fixed in space and let the unit mass move to various locations $P(x, y, z)$. The potential function ϕ is then a function of x, y, and z. If we let P move along a direction parallel to the x axis, then

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial x} \\ &= -\frac{km}{r^2}\frac{1}{2}(2)(x - X)[(x - X)^2 + (y - Y)^2 + (z - Z)^2]^{-1/2} \\ &= -\frac{km}{r^2}\frac{x - X}{r} \\ &= F\cos\alpha = F_x \end{aligned} \tag{6.9.5}$$

where α is the angle between r and the x axis, and F_x is the projection of F in the x direction. Similarly, for the other two directions,

$$F_y = \frac{\partial \phi}{\partial y}, \qquad F_z = \frac{\partial \phi}{\partial z} \tag{6.9.6}$$

The discussion above is now extended to include a distributed mass throughout a volume V. The potential $d\phi$ due to an incremental mass dm is written, following Eq. 6.9.3, as

$$d\phi = \frac{k\rho\, dV}{r} \tag{6.9.7}$$

where ρ is the density, i.e., mass per unit volume. Letting $dV = dx\, dy\, dz$, we have

$$\phi = k\iiint_V \frac{\rho\, dx\, dy\, dz}{[(x-X)^2 + (y-Y)^2 + (z-Z)^2]^{1/2}} \tag{6.9.8}$$

This is differentiated to give the force components. For example, F_x is given by

$$F_x = \frac{\partial \phi}{\partial x} = -k\iiint_V \frac{x-X}{r}\frac{\rho}{r^2}\, dx\, dy\, dz \tag{6.9.9}$$

This represents the x component of the total force exerted on a unit mass located outside the volume V at $P(x, y, z)$ due to the distributed mass in the volume V.

If we now differentiate Eq. 6.9.9 again with respect to x, we find that

$$\frac{\partial^2 \phi}{\partial x^2} = -k\iiint_V \left[\frac{1}{r^3} - \frac{3(x-X)^2}{r^5}\right]\rho\, dx\, dy\, dz \tag{6.9.10}$$

We can also show that

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial y^2} &= -k\iiint_V \left[\frac{1}{r^3} - \frac{3(y-Y)^2}{r^5}\right]\rho\, dx\, dy\, dz \\
\frac{\partial^2 \phi}{\partial z^2} &= -k\iiint_V \left[\frac{1}{r^3} - \frac{3(z-Z)^2}{r^5}\right]\rho\, dx\, dy\, dz
\end{aligned} \tag{6.9.11}$$

The sum of the bracketed terms inside the three integrals above is observed to be identically zero, using Eq. 6.9.2. Hence, Laplace's equation results,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{6.9.12}$$

or, in our shorthand notation,

$$\nabla^2 \phi = 0 \tag{6.9.13}$$

Laplace's equation is also satisfied by a magnetic potential function and an electric potential function at points not occupied by magnetic poles or electric charges. We have already observed in Section 6.5 that the steady-state heat-conduction problem leads to Laplace's equation. Finally, the flow of an incompressible fluid with negligible viscous effects also leads to Laplace's equation.

We have now derived several partial differential equations that describe a variety of physical phenomena. This modeling process is quite difficult to perform on a situation that is new and different. Hopefully, the confidence gained in deriving the equations of this chapter and in finding solutions will allow the reader to derive and solve other partial differential equations arising in the multitude of application areas.

PROBLEMS

6.1 Classify each of the following equations.

(a) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial x\,\partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(b) $(1-x)\dfrac{\partial^2 u}{\partial x^2} + 2y\dfrac{\partial^2 u}{\partial x\,\partial y} + (1+x)\dfrac{\partial^2 u}{\partial y^2} = 0$

(c) $\dfrac{\partial^2 u}{\partial x^2} + \sqrt{1+\left(\dfrac{\partial u}{\partial x}\right)^2}\,\dfrac{\partial^2 u}{\partial y^2} + k\dfrac{\partial u}{\partial y} = G(x, y)$

(d) $\left(\dfrac{\partial u}{\partial x}\right)^2 = u(x, y)$

(e) $\dfrac{du}{dx} = u(x)$

6.2 Verify each of the following statements.

(a) $u(x, y) = e^x \sin y$ is a solution of Laplace's equation, $\nabla^2 u = 0$.

(b) $T(x, t) = e^{-kt} \sin x$ is a solution of the parabolic heat equation, $\partial T/\partial t = k\partial^2 T/\partial x^2$.

(c) $u(x, t) = \sin \omega x \sin \omega a t$ is a solution of the wave equation, $\partial^2 u/\partial t^2 = a^2 \partial^2 u/\partial x^2$.

6.3 In arriving at the equation describing the motion of a vibrating string, the weight was assumed to be negligible. Include the weight of the string in the derivation and determine the describing equation. Classify the equation.

6.4 Derive the describing equation for a stretched string subject to gravity loading and viscous drag. Viscous drag per unit length of string may be expressed by $c(\partial u/\partial t)$; the drag force is proportional to the velocity. Classify the resulting equation.

6.5 Derive the torsional vibration equation for a circular shaft by applying the basic law which states that $I\alpha = \sum T$, where α is the angular acceleration, T is the torque ($T = GJ\theta/L$, where θ is the angle of twist of the shaft of length L and J and G are constants), and I is the mass moment of inertia ($I = k^2 m$, where the radius of gyration $k = \sqrt{J/A}$ and m is the mass of the shaft). Choose an infinitesimal element of the shaft of length Δx, sum the torques acting on it, and, using ρ as the mass density, show that the wave equation results,

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{G}{\rho}\frac{\partial^2 \theta}{\partial x^2}$$

6.6 An unloaded beam will undergo vibrations when subjected to an initial disturbance. Derive the appropriate partial differential equation which describes the motion using Newton's second law applied to an infinitesimal section of the beam. Assume the inertial force to be a distributed load acting on the beam. A uniformly distributed load w is related to the vertical deflection $y(x, t)$ of the beam by $w = -EI\,\partial^4 y/\partial x^4$, where E and I are constants.

6.7 For the special situation in which $LG = RC$, show that the transmission-line equation 6.2.36 reduces to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

if we let

$$i(x, t) = e^{-abt} u(x, t)$$

where $a^2 = 1/LC$ and $b^2 = RG$.

6.8 A tightly stretched string, with its ends fixed at the points (0, 0) and (2L, 0), hangs at rest under its own weight. The y axis points vertically upward. Find the describing equation for the position $u(x)$ of the string. Is the following expression a solution?

$$u(x) = \frac{g}{2a^2}(x - L)^2 - \frac{gL^2}{2a^2}$$

where $a^2 = P/m$. If so, show that the depth of the vertex of the parabola (i.e., the lowest point) varies directly with m (mass per unit length) and L^2, and inversely with P, the tension.

6.9 A very long string is given an initial displacement $\phi(x)$ and an initial velocity $\theta(x)$. Determine the general form of the solution for $u(x, t)$. Compare with the solution (6.3.18) and that of Example 6.1.

6.10 An infinite string with a mass of 0.03 kg/m is stretched with a force of 300 N. It is subjected to an initial displacement of $\cos x$ for $-\pi/2 < x < \pi/2$ and zero for all other x and released from rest. Determine the subsequent displacement of the string and sketch the solution for $t = 0.1$ s and 0.01 s.

6.11 Express the solution (6.4.36) in terms of the solution (6.3.10). What are f and g?

6.12 Determine the general solution for the wave equation using separation of variables assuming that the separation constant is zero. Show that this solution cannot satisfy the boundary and/or initial conditions.

6.13 Verify that

$$u(x, t) = b_n \cos\frac{\pi a t}{L} \sin\frac{n\pi x}{L}$$

is a solution to Eq. 6.4.1, and the conditions 6.4.2 through 6.4.4.

6.14 Find the constants A, B, C, and D in Eqs. 6.4.23 and 6.4.24 in terms of the constants c_1, c_2, c_3, and c_4 in Eqs. 6.4.20 and 6.4.21.

6.15 Determine the relationship of the fundamental frequency of a vibrating string to the mass per unit length, the length of the string, and the tension in the string.

6.16 If, for a vibrating wire, the original displacement of the 2-m-long stationary wire is given by a) $0.1 \sin x\pi/2$, b) $0.1 \sin 3\pi/2$, and c) $0.1(\sin \pi x/2 - \sin 3\pi x/2)$, find the displacement function $u(x, t)$. Both ends are fixed, $P = 50$ N, and the mass per unit length is 0.01 kg/m. With what frequency does the wire oscillate? Write the eigenvalue and eigenfunction for part (a).

6.17 The initial displacement in a 2-m-long string is given by $0.2 \sin \pi x$ and released from rest. Calculate the maximum velocity in the string and state its location.

6.18 A string π m long is stretched until the wave speed is 40 m/s. It is given an initial velocity of $4 \sin x$ from its equilibrium position. Determine the maximum displacement and state its location and when it occurs.

6.19 A string 4 m long is stretched, resulting in a wave speed of 60 m/s. It is given an initial displacement of $0.2 \sin \pi x/4$ and an initial velocity of $20 \sin \pi x/4$. Find the solution representing the displacement of the string.

6.20 A 4-m-long stretched string with $a = 20$ m/s is fixed at each end.

(a) The string is started off by an initial displacement $u(x, 0) = 0.2 \sin \pi x/4$. The initial velocity is zero. Determine the solution for $u(x, t)$.

(b) Suppose that we wish to generate the same string vibration as in part (a) (a standing half-sine wave with the same amplitude), but we want to start with a zero-displacement, non-zero-velocity condition. That is, $u(x, 0) = 0$, $\partial u/\partial t(x, 0) = g(x)$. What should $g(x)$ be?

(c) For $u(x, 0) = 0.1 \sin \pi x/4$ and $\partial u/\partial t(x, 0) = 10 \sin \pi x/4$, what are the arbitrary constants? What is the maximum displacement value $u_{\max}(x, t)$, and where does it occur?

6.21 Suppose that a tight string is subjected to the following conditions: $u(0, t) = 0$, $u(L, t) = 0$, $\partial u/\partial t(x, 0) = 0$, $u(x, 0) = k$. Calculate the first three nonzero terms of the solution $u(x, t)$.

6.22 A string π m long is started into motion by giving the middle one-half an initial velocity of 20 m/s. The string is stretched until the wave speed is 60 m/s. Determine the resulting displacement of the string as a function of x and t.

6.23 The right end of a 6-m-long wire, which is stretched until the wave speed is 60 m/s, is continually moved with the displacement $0.5 \cos 4\pi t$. What is the maximum amplitude of the resulting displacement?

6.24 The wind is blowing over some suspension cables on a bridge, causing a force that is approximated by the function $0.02 \sin 21\pi t$. Is resonance possible if the force in the cable is 40,000 N, the cable has a mass of 10 kg/m, and it is 15 m long?

6.25 A circular shaft π m long is fixed at both ends. The middle of the shaft is twisted through an angle α, the remainder of the shaft through an angle proportional to the distance from the nearest end, and then the shaft is released from rest. Determine the subsequent motion expressed as $\theta(x, t)$. Problem 6.5 gives the appropriate wave equation.

6.26 Modify Eq. 6.5.9 to account for internal heat generation within the rod. The rate of heat generation is denoted ϕ (W/m$^3 \cdot$ s).

6.27 Allow the sides of a long, slender circular rod to transfer heat by convection. The convective rate of heat loss is given by $Q = hA(T - T_f)$, where h (W/m$^2 \cdot$ s $\cdot$ °C) is the convection coefficient, A is the surface area, and T_f is the temperature of the surrounding fluid. Derive the describing partial differential equation. (*Hint:* Apply an energy balance to an elemental slice of the rod.)

6.28 The tip of a 2-m-long slender rod with lateral surface insulated is dipped into a hot liquid at 200°C. What differential equation would describe the temperature? After a long time, what would be the temperature distribution in the rod if the other end is held at 0°C? The lateral surfaces of the rod are insulated.

6.29 The conductivity K in the derivation of Eq. 6.3.10 was assumed constant. Let K be a function of x and let C and ρ be constants. Write the appropriate describing equation.

6.30 Write the one-dimensional heat equation that would be used to determine the temperature in a) a flat circular disk with the flat surfaces insulated, and b) in a sphere with initial temperature a function of r only.

6.31 Determine the steady-state temperature distribution in a) a flat circular disc with sides held at 100°C with the flat surfaces insulated, and b) a sphere with the outer surface held at 100°C.

6.32 The initial temperature in a 10-m-long iron rod is 300 sin $\pi x/10$, with both ends being held at zero temperature. Determine the times necessary for the midpoint of the rod to reach 200, 100, and 50, respectively. The material constant $k = 1.7 \times 10^{-5}$ m^2/s. The lateral surfaces are insulated.

6.33 A 1-m-long, 50-mm-diameter aluminum rod, with lateral surfaces insulated, is initially at a temperature of $200(1 + \sin \pi x)$. Calculate the rate at which the rod is transferring heat at the left end initially and after 600 s if both ends are maintained at 200°C. For aluminum, $K = 200$ W/m $\cdot$ °C and $k = 8.6 \times 10^{-5}$ m^2/s. (*Hint:* Let $\theta(x, t) = T(x, t) - 200$.)

6.34 The initial temperature distribution in a 2-m-long brass bar is given by

$$f(x) = \begin{cases} 50x & 0 < x < 1 \\ 100 - 50x & 1 < x < 2 \end{cases}$$

Both ends are maintained at zero temperature. Determine the solution for $T(x, t)$. How long would you predict it would take for the center of the rod to reach a temperature of 10°C? The material constant $k = 2.9 \times 10^{-5}$ m^2/s. The lateral surfaces are insulated.

6.35 The initial temperature distribution in a 2-m-long steel rod is given by

$$f(x) = \begin{cases} 50x & 0 < x < 1 \\ 100 - 50x & 1 < x < 2 \end{cases}$$

The rod is completely insulated. Determine the temperature distribution in the rod and predict the temperature that the rod will eventually attain. $k = 3.9 \times 10^{-6}$ m^2/s.

6.36 A 2-m-long aluminum bar, with lateral surfaces insulated, is given the initial temperature distribution $f(x) = 50\,x^2$. The left end of the bar is maintained at 0°C and the right end at 200°C. Determine the subsequent temperature distribution in the bar. $k = 8.6 \times 10^{-5}$ m^2/s.

6.37 The infinite slab of Fig. 6.17 is initially at temperature $f(x)$. The face at $x = 0$ is held at $T = 0$°C. Determine the temperature $T(x, t)$ of the slab for $t > 0$.

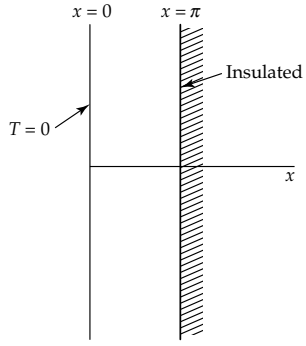


FIGURE 6.17

6.38 The aluminum slab in Problem 6.37 is given the initial temperature distribution

$$f(x) = \begin{cases} 100 & 0 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$$

Estimate the rate of heat transfer per square meter from the left face at $t = 10^4$ s if $k = 8.6 \times 10^{-5}$ m^2/s and $K = 200$ W/m · °C.

6.39 Heat generation occurs within a 4-m-long copper rod at the variable rate of $2000(4x - x^2)$ W/m^3 · s. Both ends are maintained at 0°C. $C = 380$ J/kg · °C, $\rho = 8940$ kg/m^3, and $k = 1.14 \times 10^{-4}$ m^2/s.

(a) Find the steady-state solution for the temperature distribution in the rod.

(b) Find the transient temperature distribution in the rod if the initial temperature was constant at 100°C. Just set up the integral for the Fourier coefficients; do not integrate.

6.40 Find the steady-state temperature distribution in a 1-m^2 slab if three sides are maintained at 0°C and the remaining side (at $y = 1$ m) is held at 100 sin πx °C. All other surfaces are insulated.

6.41 Three edges of a thin 1-m by 2-m plate are held at 0°C, while the fourth edge, at $y = 1$ m, is held at 100°C. All other surfaces are insulated. Determine an expression for the temperature distribution in the plate.

6.42 Find the steady-state temperature distribution in a 2 m-square slab if three sides are maintained at 100°C and the remaining side (at $x = 2$ m) is held at 200°C. The two flat surfaces are insulated.

6.43 The temperature of a spherical surface 0.2 m in diameter is maintained at a temperature of 250°C. This surface is interior to a very large mass. Find an expression for the temperature distribution inside and outside the surface.

6.44 The temperature on the surface of a 1-m-diameter sphere is 100 cos ϕ °C. What is the temperature distribution inside the sphere?

6.45 Find the potential field between two concentric spheres if the potential of the outer sphere is maintained at $V = 100$ and the potential of the inner sphere is maintained at zero. The radii are 2 m and 1 m, respectively.

6.46 A right circular cylinder is 1 m long and 2 m in diameter. Its left end and lateral surface are maintained at a temperature of 0°C and its right end at 100°C. Find an expression for its temperature at any interior point. Calculate the first three coefficients in the series expansion.

6.47 Determine the solution for the temperature as a function of r and t in a circular cylinder of radius r_0 with insulated (or infinitely long) ends if the initial temperature distribution is a function $f(r)$ of r only and the lateral surface is maintained at 0°C. See Eq. 6.5.14.

6.48 An aluminum circular cylinder 50 mm in diameter with ends insulated is initially at 100°C. Approximate the temperature at the center of the cylinder after 2 s if the lateral surface is kept at 0°C. For aluminum, $k = 8.6 \times 10^{-5}$ m^2/s.

6.49 A circular cylinder 1 m in radius is completely insulated and has an initial temperature distribution l00r °C. Find an expression for the temperature as a function of r and t. Write integral expressions for at least three coefficients in the series expansion.

6.50 Differentiate Eq. 6.9.8 and show that Eq. 6.9.9 results. Also verify Eq. 6.9.10.