SADDLE POINT

The second case (say $\lambda_1 < 0$ and $\lambda_2 > 0$) corresponds to a *saddle point* (Figure 2.9(c)). The phase portrait of the system has the interesting "saddle" shape shown in Figure 2.9(c). Because of the unstable pole λ_2, almost all of the system trajectories diverge to infinity. In this figure, one also observes two straight lines passing through the origin. The diverging line (with arrows pointing to infinity) corresponds to initial conditions which make k_2 (*i.e.*, the unstable component) equal zero. The converging straight line corresponds to initial conditions which make k_1 equal zero.

STABLE OR UNSTABLE FOCUS

The third case corresponds to a focus. A *stable focus* occurs when the real part of the eigenvalues is negative, which implies that $x(t)$ and $\dot{x}(t)$ both converge to zero. The system trajectories in the vicinity of a stable focus are depicted in Figure 2.9(d). Note that the trajectories encircle the origin one or more times before converging to it, unlike the situation for a stable node. If the real part of the eigenvalues is positive, then $x(t)$ and $\dot{x}(t)$ both diverge to infinity, and the singularity point is called an *unstable focus*. The trajectories corresponding to an unstable focus are sketched in Figure 2.9(e).

CENTER POINT

The last case corresponds to a center point, as shown in Figure 2.9(f). The name comes from the fact that all trajectories are ellipses and the singularity point is the center of these ellipses. The phase portrait of the undamped mass-spring system belongs to this category.

Note that the stability characteristics of linear systems are uniquely determined by the nature of their singularity points. This, however, is not true for nonlinear systems.

2.5 Phase Plane Analysis of Nonlinear Systems

In discussing the phase plane analysis of nonlinear systems, two points should be kept in mind. Phase plane analysis of nonlinear systems is related to that of linear systems, because the local behavior of a nonlinear system can be approximated by the behavior of a linear system. Yet, nonlinear systems can display much more complicated patterns in the phase plane, such as multiple equilibrium points and limit cycles. We now discuss these points in more detail.

LOCAL BEHAVIOR OF NONLINEAR SYSTEMS

In the phase portrait of Figure 2.2, one notes that, in contrast to linear systems, there are two singular points, $(0,0)$ and $(-3,0)$. However, we also note that the features of the phase trajectories in the neighborhood of the two singular points look very much like those of linear systems, with the first point corresponding to a stable focus and the second to a saddle point. This similarity to a linear system in the local region of each singular point can be formalized by linearizing the nonlinear system, as we now discuss.

If the singular point of interest is not at the origin, by defining the difference between the original state and the singular point as a new set of state variables, one can always shift the singular point to the origin. Therefore, without loss of generality, we may simply consider Equation (2.1) with a singular point at 0. Using Taylor expansion, Equations (2.1a) and (2.1b) can be rewritten as

$$\dot{x}_1 = a x_1 + b x_2 + g_1(x_1, x_2)$$

$$\dot{x}_2 = c x_1 + d x_2 + g_2(x_1, x_2)$$

where g_1 and g_2 contain higher order terms.

In the vicinity of the origin, the higher order terms can be neglected, and therefore, the nonlinear system trajectories essentially satisfy the linearized equation

$$\dot{x}_1 = a\, x_1 + b\, x_2$$

$$\dot{x}_2 = c\, x_1 + d\, x_2$$

As a result, the local behavior of the nonlinear system can be approximated by the patterns shown in Figure 2.9.

LIMIT CYCLES

In the phase portrait of the nonlinear Van der Pol equation, shown in Figure 2.8, one observes that the system has an unstable node at the origin. Furthermore, there is a closed curve in the phase portrait. Trajectories inside the curve and those outside the curve all tend to this curve, while a motion started on this curve will stay on it forever, circling periodically around the origin. This curve is an instance of the so-called "limit cycle" phenomenon. Limit cycles are unique features of nonlinear systems.

In the phase plane, *a limit cycle is defined as an isolated closed curve*. The trajectory has to be both closed, indicating the periodic nature of the motion, and isolated, indicating the limiting nature of the cycle (with nearby trajectories converging or diverging from it). Thus, while there are many closed curves in the phase portraits of the mass-spring-damper system in Example 2.1 or the satellite system in Example 2.5, these are not considered limit cycles in this definition, because they are not isolated.

Depending on the motion patterns of the trajectories in the vicinity of the limit cycle, one can distinguish three kinds of limit cycles

1. **Stable Limit Cycles**: all trajectories in the vicinity of the limit cycle converge to it as $t \to \infty$ (Figure 2.10(a));
2. **Unstable Limit Cycles**: all trajectories in the vicinity of the limit cycle diverge from it as $t \to \infty$ (Figure 2.10(b));
3. **Semi-Stable Limit Cycles**: some of the trajectories in the vicinity converge to it, while the others diverge from it as $t \to \infty$ (Figure 2.10(c));

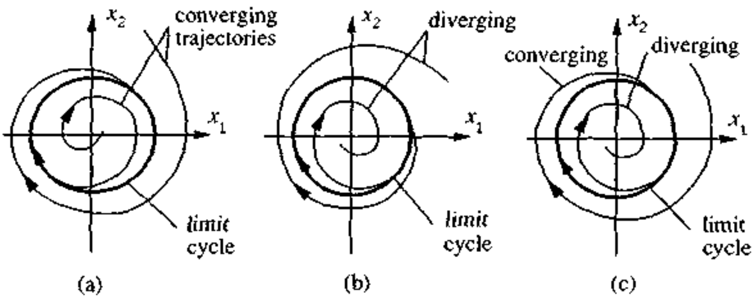

Figure 2.10 : Stable, unstable, and semi-stable limit cycles

As seen from the phase portrait of Figure 2.8, the limit cycle of the Van der Pol equation is clearly stable. Let us consider some additional examples of stable, unstable, and semi-stable limit cycles.

Example 2.7: stable, unstable, and semi-stable limit cycles

Consider the following nonlinear systems

(a) $\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1) \qquad \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)$ (2.12)

(b) $\dot{x}_1 = x_2 + x_1(x_1^2 + x_2^2 - 1) \qquad \dot{x}_2 = -x_1 + x_2(x_1^2 + x_2^2 - 1)$ (2.13)

(c) $\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1)^2 \qquad \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1)^2$ (2.14)

Let us study system (a) first. By introducing polar coordinates

$$r = (x_1^2 + x_2^2)^{1/2} \qquad \theta = \tan^{-1}(x_2/x_1)$$

the dynamic equations (2.12) are transformed as

$$\frac{dr}{dt} = -r(r^2-1) \qquad \frac{d\theta}{dt} = -1$$

When the state starts on the unit circle, the above equation shows that $\dot{r}(t) = 0$. Therefore, the state will circle around the origin with a period $1/2\pi$. When $r < 1$, then $\dot{r} > 0$. This implies that the state tends to the circle from inside. When $r > 1$, then $\dot{r} < 0$. This implies that the state tends toward the unit circle from outside. Therefore, the unit circle is a stable limit cycle. This can also be concluded by examining the analytical solution of (2.12)

$$r(t) = \frac{1}{(1 + c_o e^{-2t})^{1/2}} \qquad \theta(t) = \theta_o - t$$

where

$$c_o = \frac{1}{r_o^2} - 1$$

Similarly, one can find that the system (b) has an unstable limit cycle and system (c) has a semi-stable limit cycle. □

2.6 Existence of Limit Cycles

As mentioned in chapter 1, it is of great importance for control engineers to predict the existence of limit cycles in control systems. In this section, we state three simple classical theorems to that effect. These theorems are easy to understand and apply.

The first theorem to be presented reveals a simple relationship between the existence of a limit cycle and the number of singular points it encloses. In the statement of the theorem, we use N to represent the number of nodes, centers, and foci enclosed by a limit cycle, and S to represent the number of enclosed saddle points.

Theorem 2.1 (Poincare) *If a limit cycle exists in the second-order autonomous system (2.1), then* $N = S + 1$.

This theorem is sometimes called the *index theorem*. Its proof is mathematically involved (actually, a family of such proofs led to the development of algebraic topology) and shall be omitted here. One simple inference from this theorem is that a limit cycle must enclose at least one equilibrium point. The theorem's result can be verified easily on Figures 2.8 and 2.10.

The second theorem is concerned with the asymptotic properties of the trajectories of second-order systems.

Theorem 2.2 (Poincare-Bendixson) *If a trajectory of the second-order autonomous system remains in a finite region* Ω*, then one of the following is true:*

(a) the trajectory goes to an equilibrium point

(b) the trajectory tends to an asymptotically stable limit cycle

(c) the trajectory is itself a limit cycle

While the proof of this theorem is also omitted here, its intuitive basis is easy to see, and can be verified on the previous phase portraits.

The third theorem provides a sufficient condition for the non-existence of limit cycles.

Theorem 2.3 (Bendixson) *For the nonlinear system (2.1), no limit cycle can exist in a region* Ω *of the phase plane in which* $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ *does not vanish and does not change sign.*

Proof: Let us prove this theorem by contradiction. First note that, from (2.5), the equation

$$f_2\, dx_1 - f_1\, dx_2 = 0 \tag{2.15}$$

is satisfied for any system trajectories, including a limit cycle. Thus, along the closed curve L of a limit cycle, we have

$$\int_L (f_1\, dx_2 - f_2\, dx_1) = 0 \tag{2.16}$$

Using Stokes' Theorem in calculus, we have

$$\int_L (f_1\, dx_2 - f_2\, dx_1) = \int\int \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1\, dx_2 = 0$$

where the integration on the right-hand side is carried out on the area enclosed by the limit cycle.

By Equation (2.16), the left-hand side must equal zero. This, however, contradicts the fact that the right-hand side cannot equal zero because by hypothesis $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ does not vanish and does not change sign. □

Let us illustrate the result on an example.

Example 2.8: Consider the nonlinear system

$$\dot{x}_1 = g(x_2) + 4x_1x_2^2$$

$$\dot{x}_2 = h(x_1) + 4x_1^2x_2$$

Since

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4(x_1^2 + x_2^2)$$

which is always strictly positive (except at the origin), the system does not have any limit cycles anywhere in the phase plane. □

The above three theorems represent very powerful results. It is important to notice, however, that they have no equivalent in higher-order systems, where exotic asymptotic behaviors other than equilibrium points and limit cycles can occur.

2.7 Summary

Phase plane analysis is a graphical method used to study second-order dynamic systems. The major advantage of the method is that it allows visual examination of the global behavior of systems. The major disadvantage is that it is mainly limited to second-order systems (although extensions to third-order systems are often achieved with the aid of computer graphics). The phenomena of multiple equilibrium points and of limit cycles are clearly seen in phase plane analysis. A number of useful classical theorems for the prediction of limit cycles in second-order systems are also presented.

2.8 Notes and References

Phase plane analysis is a very classical topic which has been addressed by numerous control texts. An extensive treatment can be found in [Graham and McRuer, 1961]. Examples 2.2 and 2.3 are adapted from [Ogata, 1970]. Examples 2.5 and 2.6 and section 2.6 are based on [Hsu and Meyer, 1968].

2.9 Exercises

2.1 Draw the phase portrait and discuss the properties of the linear, unity feedback control system of open-loop transfer function

$$G(p) = \frac{10}{p\,(1 + 0.1\,p)}$$

2.2 Draw the phase portraits of the following systems, using isoclines

(*a*) $\ddot{\theta} + \dot{\theta} + 0.5\,\theta = 0$

(*b*) $\ddot{\theta} + \dot{\theta} + 0.5\,\theta = 1$

(*c*) $\ddot{\theta} + \dot{\theta}^2 + 0.5\,\theta = 0$

2.3 Consider the nonlinear system

$$\dot{x} = y + x\,(x^2 + y^2 - 1)\sin\frac{1}{x^2 + y^2 - 1}$$

$$\dot{y} = -x + y\,(x^2 + y^2 - 1)\sin\frac{1}{x^2 + y^2 - 1}$$

Without solving the above equations explicitly, show that the system has infinite number of limit cycles. Determine the stability of these limit cycles. (*Hint:* Use polar coordinates.)

2.4 The system shown in Figure 2.10 represents a satellite control system with rate feedback provided by a gyroscope. Draw the phase portrait of the system, and determine the system's stability.

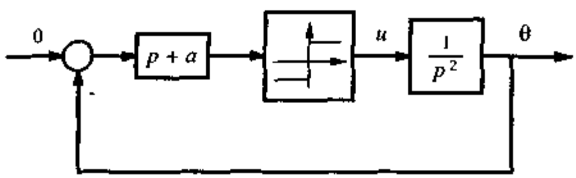

Figure 2.10 : Satellite control system with rate feedback